Differential Geometry in Equivariant CNNs via Biprincipal Bundles

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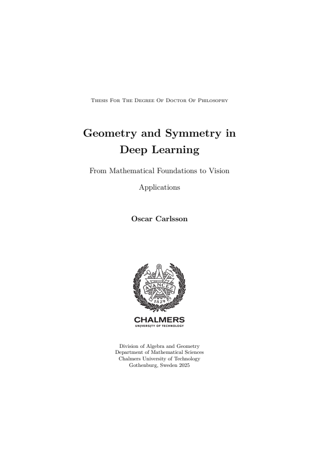
Material

PhD defence on the
29:th of August at 13:15


THESIS FOR THE DEGREE OF DOCTOR OF PHILOSOPHY

Geometry and Symmetry in
Deep Learning

From Mathematical Foundations to Vision
Applications

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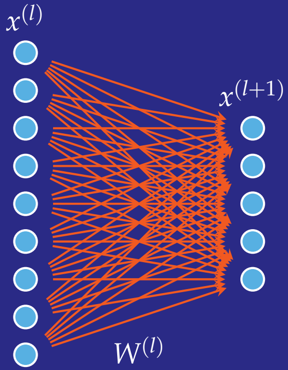
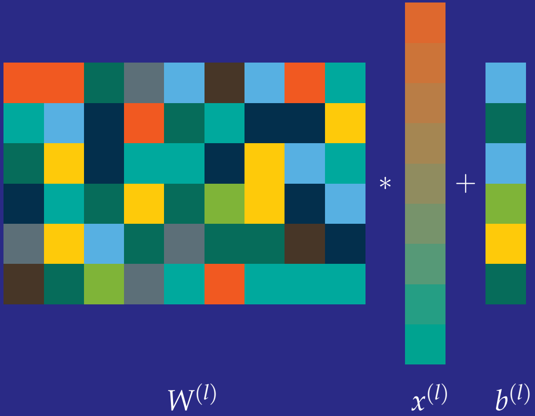
Outline

1. Introduction to the classical description of a CNN
2. CNN through differential geometry
3. Equivariant CNNs and biprincipal bundles
4. Connections to other frameworks

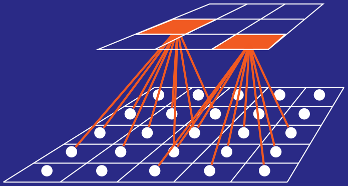
Machine learning: *fully connected* layer

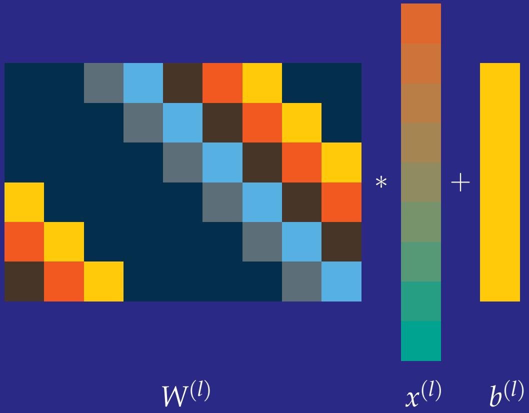




$$W^{(l)}[T_s x^{(l)}] + b^{(l)} \neq T_s[W^{(l)} x^{(l)} + b^{(l)}]$$

Machine learning: *convolutional* layer





$$W^{(l)}[T_s x^{(l)}] + b^{(l)} = T_s[W^{(l)} x^{(l)} + b^{(l)}]$$

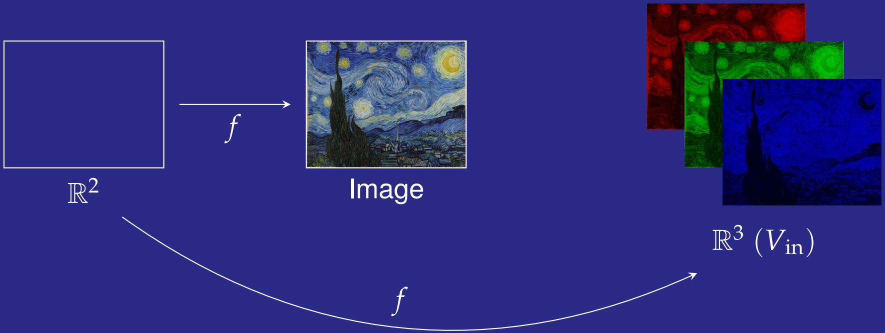
G-equivariance

A map $\phi : \mathcal{F}_{\text{in}} \to \mathcal{F}_{\text{out}}$ is G-equivariant iff

$$\phi(L_g^{\text{in}} f) = L_g^{\text{out}}[\phi(f)], \qquad \forall g \in G.$$

$$\begin{tikzcd}
\mathcal{F}_{\text{in}} \arrow[r, "\phi"] \arrow[d, "L_g"] & \mathcal{F}_{\text{out}} \arrow[d, "L_g"] \\
\mathcal{F}_{\text{in}} \arrow[r, "\phi"] & \mathcal{F}_{\text{out}}
\end{tikzcd}$$

Image as a feature map



Group of translations acting on feature maps

With $f : \mathbb{R}^2 \to V_{\text{in}}$ a translation $z \in \mathbb{R}^2$ act on f through

$$[L_z f](x) = f(L_{-z}(x)) = f(x - z)$$

Mathematics: the convolutional map

$$[\phi f](x) = \int_{\mathbb{R}^2} \widehat{\kappa}(y-x) f(y) \mathrm{d}y,$$

with $\widehat{\kappa} : \mathbb{R}^2 \to \mathrm{Hom}(V_{\mathrm{in}}, V_{\mathrm{out}})$ where

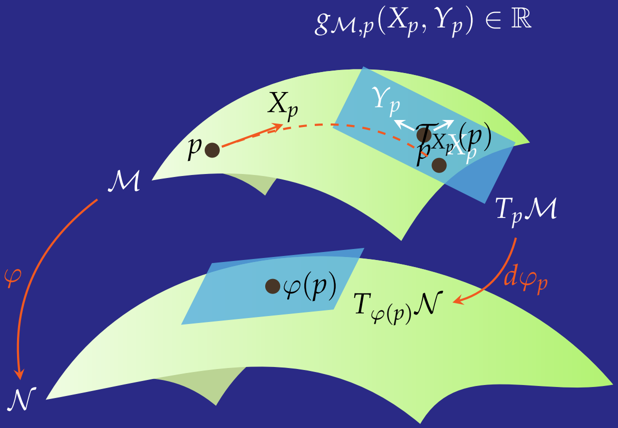
$$\widehat{\kappa}(y-x) := \kappa(0, y-x),$$

due to weight sharing:

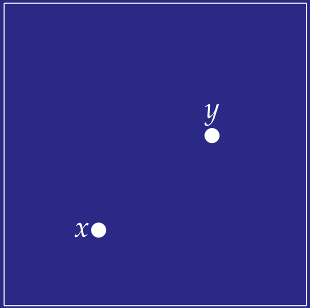
$$\kappa(x,y) = \kappa(x-z, y-z), \quad \forall z \in \mathbb{R}^2.$$

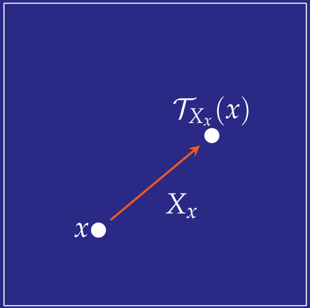
Key concepts from differential geometry

- (Smooth) Manifold (Locally $\mathbb{R}^n$)
- Tangent space ($T_p\mathcal{M} \cong \mathbb{R}^n$)
- Metric (Inner product)
- Diffeomorphism (Smooth deformations)
- Differential
- (Geodesic) Flow



Translating the general kernel





$$\kappa : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathrm{Hom}(V_{\mathrm{in}}, V_{\mathrm{out}})$$

$$\kappa : \mathbb{R}^2 \times T\mathbb{R}^2 \to \mathrm{Hom}(V_{\mathrm{in}}, V_{\mathrm{out}})$$

General map

Normal approach

$$[\phi f](x) = \int_{\mathbb{R}^2} \kappa(x, y) f(y) \mathrm{d}y$$

Differential geometry

$$[\phi f](x) = \int_{T_x\mathbb{R}^2} \kappa(x, X_x) f(\mathcal{T}_{X_x}(x)) \mathrm{d}X_x$$

Equivariance and weight sharing

Assuming translation equivariance $[\phi[L_z f]](x) = [L_z[\phi f]](x)$ yields weight sharing as

Normal approach

$$\kappa(x, y) = \kappa(x - z, y - z)$$

Differential geometry

$$\kappa(x, X_x) = \kappa\big(L_z(x), (dL_z)_x(X_x)\big)$$

$$\forall z \in \mathbb{R}^2.$$

Commutativity of flow and affine transformations

$$
\begin{tikzcd}
L_z(x) \arrow[r, "(dL_z)_x(X_x)"'] & \mathcal{T}_{(dL_z)_x(X_x)}(L_z(x)) \\
x \arrow[u, "L_z"] \arrow[r, "X_x"'] & \mathcal{T}_{X_x}(x) \arrow[u, "L_z"]
\end{tikzcd}
$$

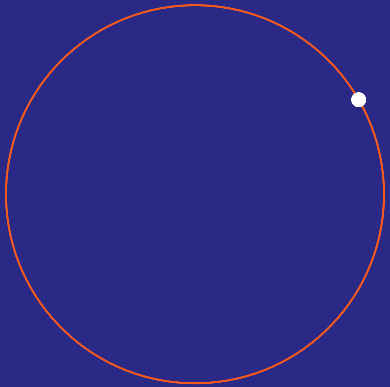
$$\mathcal{T}_{(dL_z)_x(X_x)}(L_z(x)) = L_z\big(\mathcal{T}_{X_x}(x)\big).$$

(Technically, L_z is an affine transformation)

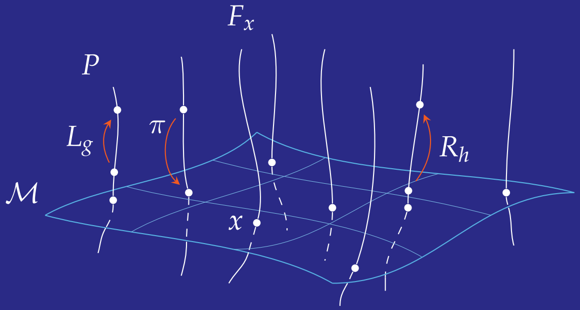
Generalisation

Smooth manifold $\mathbb{R}^2$ $\longrightarrow$ biprincipal bundle P

Group vs TorsorGroup vs *Torsor*



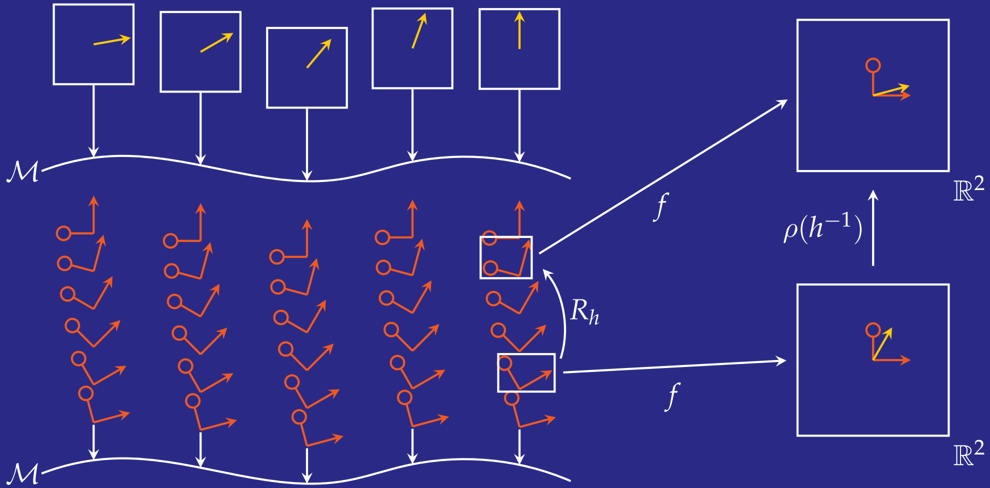
Principal H-Biprincipal (G, H)–bundle



$$G \cong F_x \cong H, \quad \forall x \in \mathcal{M}, \qquad (F_x \text{ is a } (G, H)\text{-bitorsor})$$

$$L_g(R_h(p)) = R_h(L_g(p))$$

Feature map



Feature map
-Mackey condition

$$f : P \to V_{\text{in}}, \qquad f(R_h(p)) = \rho_{\text{in}}(h^{-1})f(p), \quad \forall h \in H.$$

Generalise the start point of Diffgeo CNN to Biprincipal

CNN through differential geometry

$$[\phi f](x) = \int_{B_R(x)} \kappa(x, X_x) f(\mathcal{T}_{X_x}(x)) \mathrm{d}X_x,$$

where $B_R(x) \subset T_x\mathbb{R}^2$ and

$$\kappa(x, X_x) = \kappa(L_z(x), (dL_z)_x(X_x)),$$

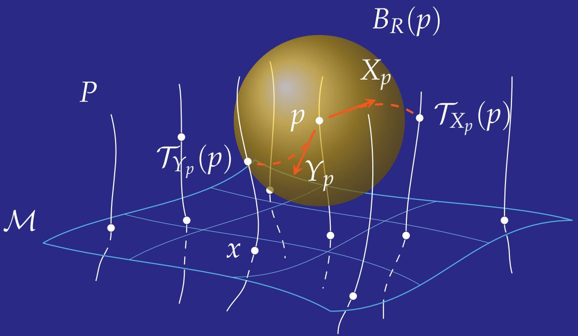
for all $z \in \mathbb{R}^2$.

CNN on biprincipal bundle

$$[\phi f](p) = \int_{B_R(p)} \kappa(p, X_p) f(\mathcal{T}_{X_p}(p)) \mathrm{d}X_p,$$

where $B_R(p) \subset T_pP$.

Visualisation of bipricipal convolution



$$[\phi f](p) = \int_{B_R(p)} \kappa(p, X_p) f(\mathcal{T}_{X_p}(p)) \mathrm{d}X_p,$$

Group action on feature maps

$$[L_g f](p) = f(L_{g^{-1}}(p)).$$

Requirements

Mackey condition:

$$[\phi f](R_h(p)) = \rho_{\text{out}}(h^{-1})[\phi f](p), \qquad \forall h \in H.$$

Equivariance:

$$[L_g[\phi f]](p) = [\phi[L_g f]](p), \qquad \forall g \in \widehat{G} \subset G.$$

Results: equivariance (weight sharing) and Mackey

Mackey condition:

$$\kappa(R_h(p), (dR_h)_p(X_p)) = \rho_{\text{out}}(h^{-1})\kappa(p, X_p)\rho_{\text{in}}(h),$$

$$[\phi f](R_h(p)) = \rho_{\text{out}}(h^{-1})[\phi f](p)$$

for all $h \in H$.

Equivariance
(Weight sharing)

$$[L_g[\phi f]](p) = [\phi[L_g f]](p)$$

$$\kappa(p, X_p)\mathbf{1}_{B_R(p)}(X_p) = \kappa(L_{g^{-1}}(p), (dL_{g^{-1}})_p(X_p)) \left|\det\left(dL_{g^{-1}}\right)_p\right| \mathbf{1}_{\widetilde{B_R}(p)}(X_p),$$

for all $g \in \widehat{G} \subset G$, compare to CNN:

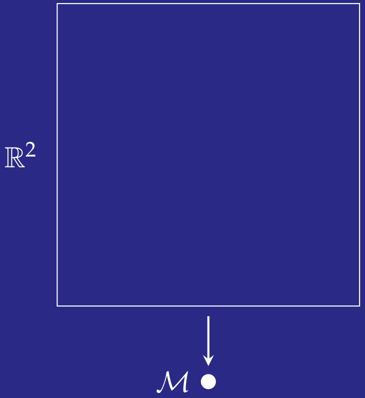
$$\kappa(x, X_p) = \kappa(L_{-z}(x), (dL_{-z})_x(X_p)), \qquad \forall z \in \mathbb{R}^2$$

where

$$\widetilde{B_R}(p) = (dL_{g^{-1}})^{-1}(B_R(L_{g^{-1}}(p))).$$

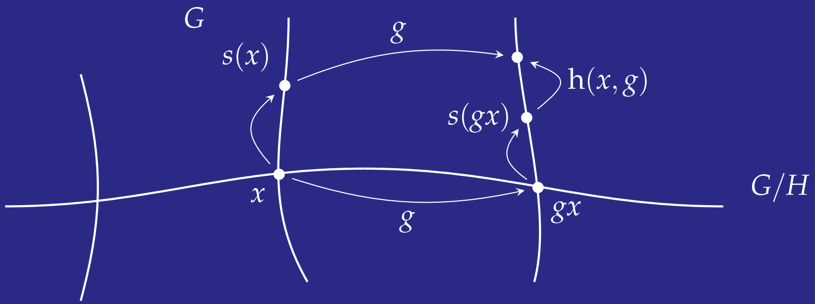
Connections to other cases

Connection to CNN



Principal homogeneous space

Connection to homogeneous case



Equivariant non-linear maps on homogeneous spaces

See Elias Nyholm's presentation

End

Thanks for your attention!

Questions?