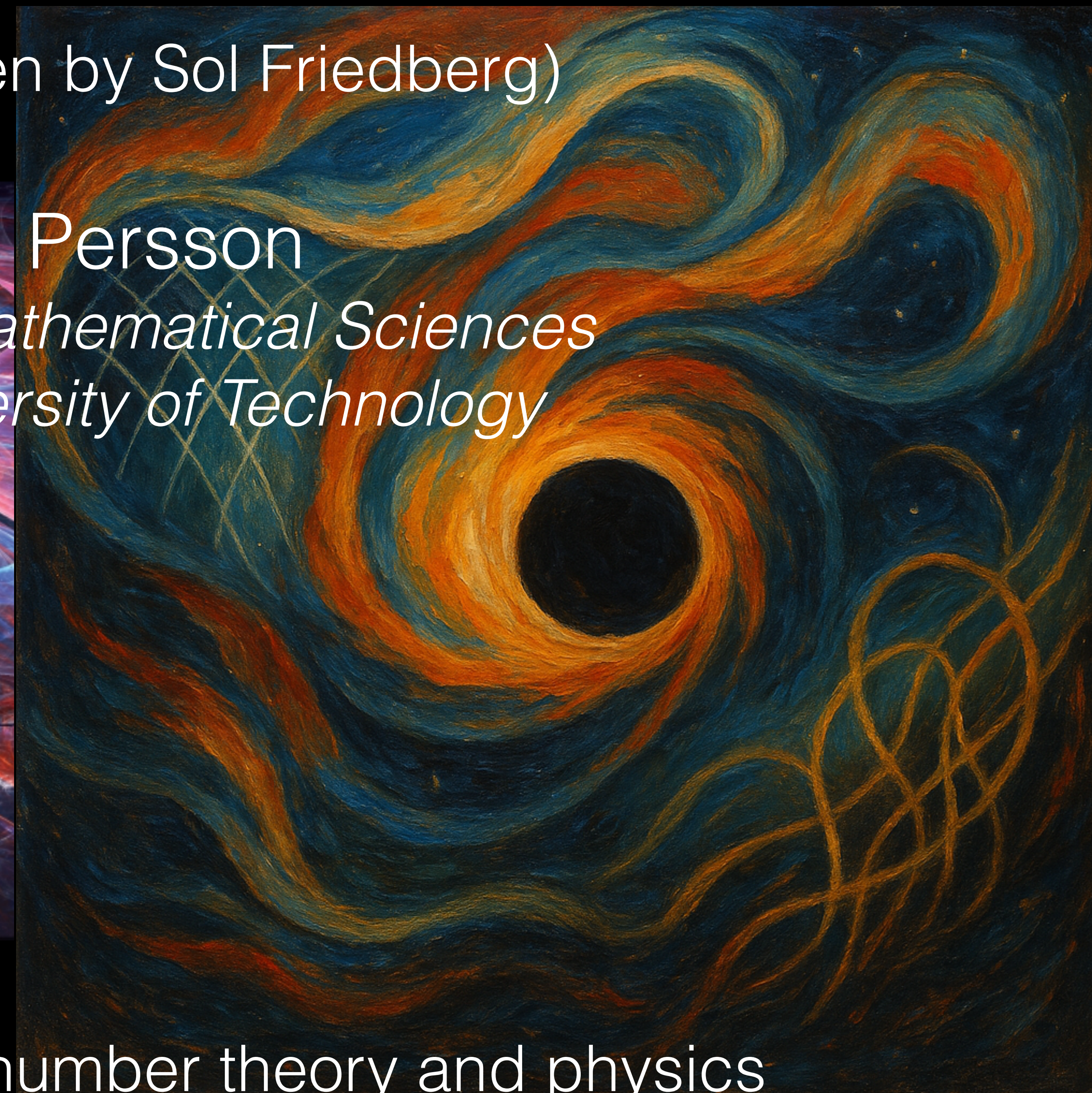
Automorphic Representations Lecture IV:
String amplitudes and automorphic representations

(First 3 lectures given by Sol Friedberg)



Daniel Persson
Department of Mathematical Sciences
Chalmers University of Technology

ICTP Winter school on number theory and physics
Trieste
November 27, 2025

Outline

1. **Background and motivation**
2. **Automorphic forms and representation theory**
3. **Fourier coefficients**
4. **Small representations**
5. **“Generalised automorphic forms”**

Suggested literature

Sol Friedberg's notes from this school:

"Automorphic forms and representations - an introduction"

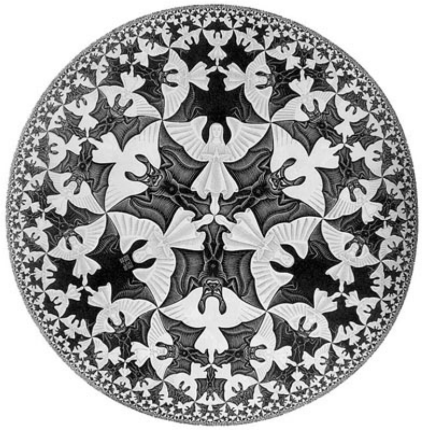
Survey: "String scattering amplitudes and small automorphic representations"
w/ Guillaume Bossard and Axel Kleinschmidt, 2024

Book: "Eisenstein series and automorphic representations - with applications in string theory"

w/ Philipp Fleig, Axel Kleinschmidt and Henrik Gustafsson
Cambridge University Press, 2018

1. Background and Motivation

Fourier coefficients of automorphic forms



- Fourier coefficients of **classical modular forms** encode deep number-theoretic information (counting points on elliptic curves etc..)

- **Moonshine:** relations with finite sporadic groups and CFT/string theory

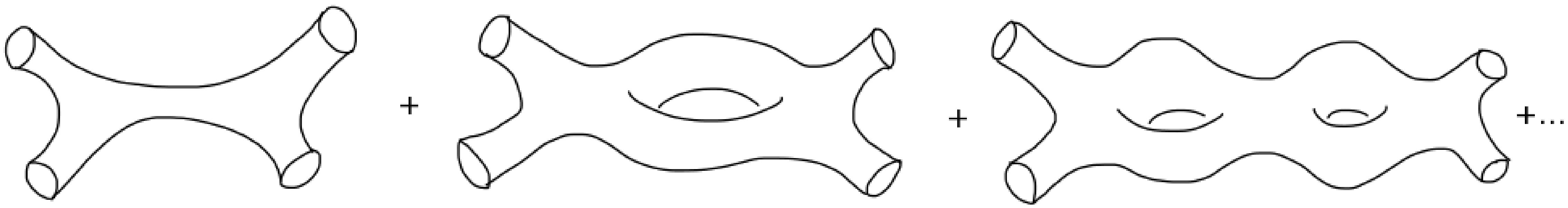
- **Enumerative geometry:** rational curves on K3, GW-theory…

- Higher rank groups: **Langlands program** (automorphic L-functions, functoriality…)

- The Fourier coefficients of Eisenstein series also encode **string theory scattering amplitudes**

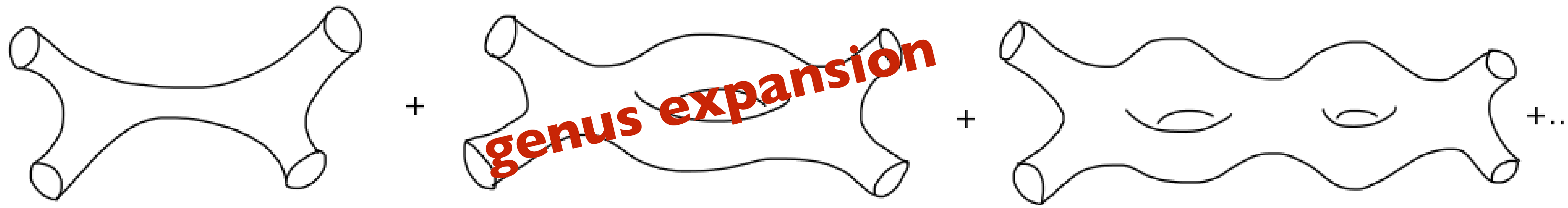
String amplitudes

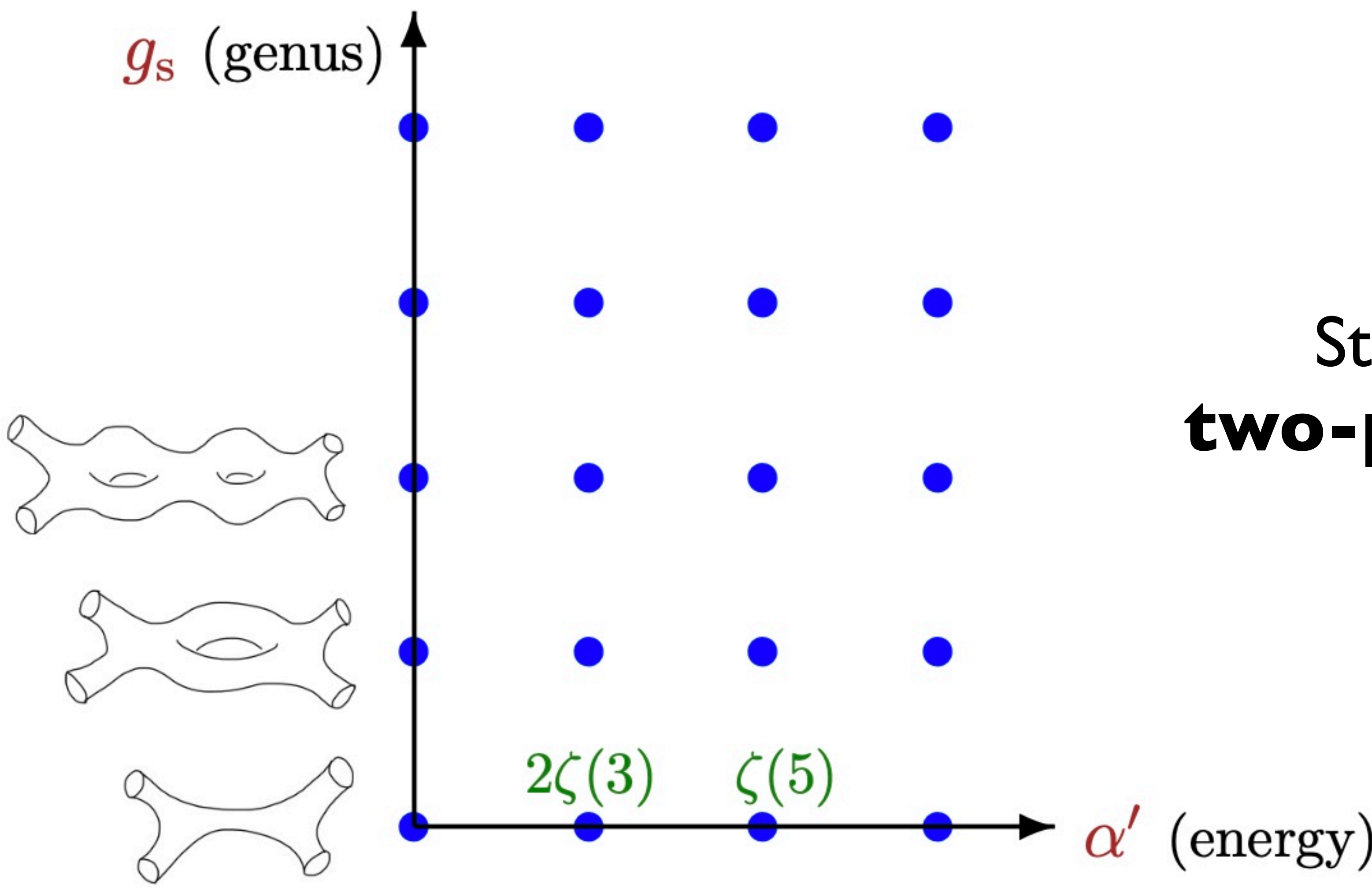
Understand the structure of **string interactions**



String amplitudes

Understand the structure of **string interactions**



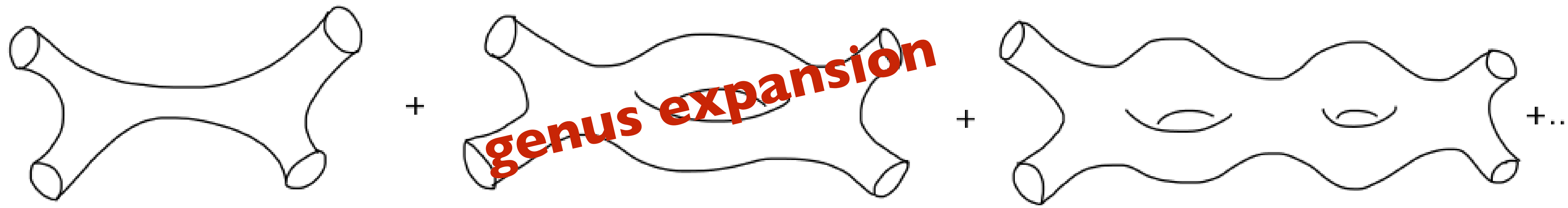


[picture stolen from Axel!]

String amplitudes have a
two-parameter expansion

String amplitudes

Understand the structure of **string interactions**



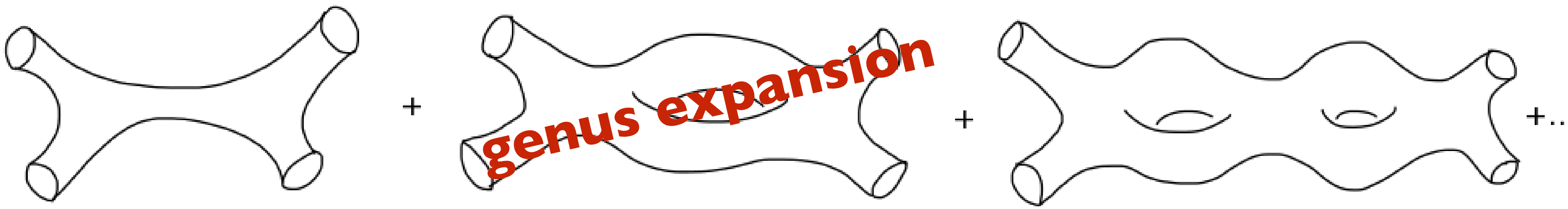
Strongly constrained by **symmetries**!

- supersymmetry
- U-duality

amplitudes have intricate **arithmetic structure** $G(\mathbb{Z})$

String amplitudes

Understand the structure of **string interactions**



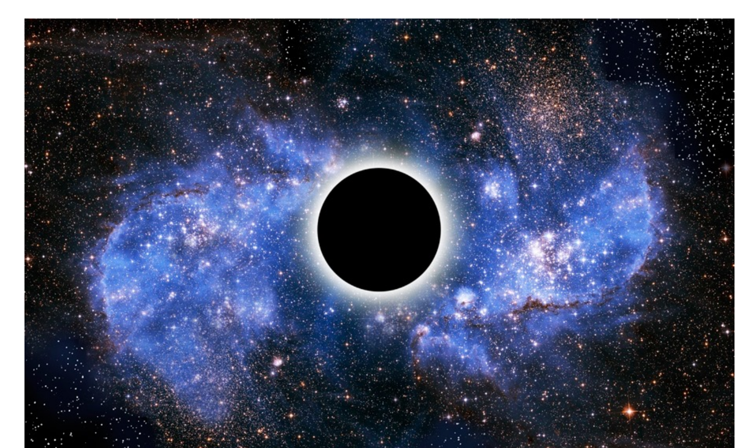
Strongly constrained by **symmetries**!

- supersymmetry
- U-duality

→ amplitudes have intricate **arithmetic structure** $G(\mathbb{Z})$

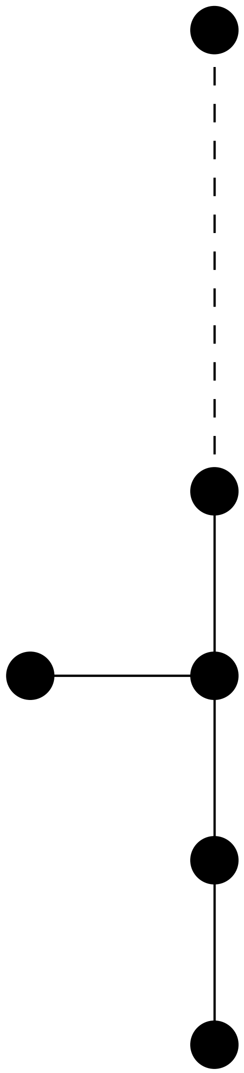
Symmetry constrains interactions, leads to insights about:

- non-perturbative effects (black holes, instantons)
- novel mathematical predictions from physics



Toroidal compactifications yield the chain of **U-duality groups**

[Cremmer, Julia][Hull, Townsend]



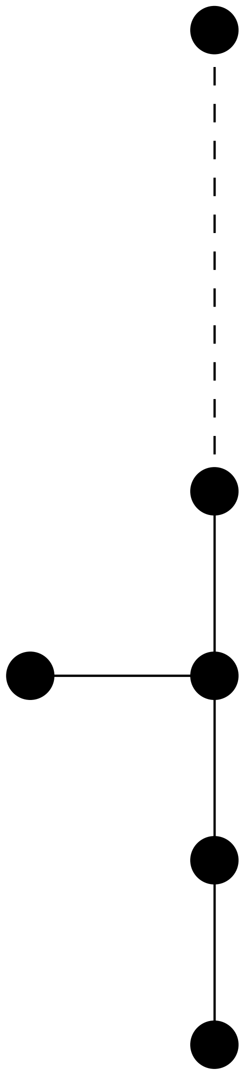
D	G	K	$G(\mathbb{Z})$
10	$\mathrm{SL}(2,\mathbb{R})$	$\mathrm{SO}(2)$	$\mathrm{SL}(2,\mathbb{Z})$
9	$\mathrm{SL}(2,\mathbb{R}) \times \mathbb{R}^+$	$\mathrm{SO}(2)$	$\mathrm{SL}(2,\mathbb{Z})$
8	$\mathrm{SL}(3,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$	$\mathrm{SO}(3) \times \mathrm{SO}(2)$	$\mathrm{SL}(3,\mathbb{Z}) \times \mathrm{SL}(2,\mathbb{Z})$
7	$\mathrm{SL}(5,\mathbb{R})$	$\mathrm{SO}(5)$	$\mathrm{SL}(5,\mathbb{Z})$
6	$\mathrm{Spin}(5,5,\mathbb{R})$	$\big(\mathrm{Spin}(5) \times \mathrm{Spin}(5)\big)/\mathbb{Z}_2$	$\mathrm{Spin}(5,5,\mathbb{Z})$
5	$\mathrm{E}_6(\mathbb{R})$	$\mathrm{USp}(8)/\mathbb{Z}_2$	$\mathrm{E}_6(\mathbb{Z})$
4	$\mathrm{E}_7(\mathbb{R})$	$\mathrm{SU}(8)/\mathbb{Z}_2$	$\mathrm{E}_7(\mathbb{Z})$
3	$\mathrm{E}_8(\mathbb{R})$	$\mathrm{Spin}(16)/\mathbb{Z}_2$	$\mathrm{E}_8(\mathbb{Z})$

Amplitudes are given by **functions** on

$$G(\mathbb{Z})\backslash G(\mathbb{R})/K$$

Toroidal compactifications yield the chain of **U-duality groups**

[Cremmer, Julia][Hull, Townsend]

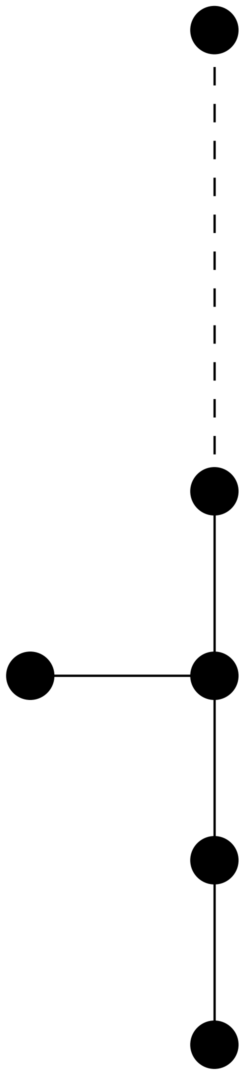


D	G	K	$G(\mathbb{Z})$
10	$\mathrm{SL}(2,\mathbb{R})$	$\mathrm{SO}(2)$	$\mathrm{SL}(2,\mathbb{Z})$
9	$\mathrm{SL}(2,\mathbb{R}) \times \mathbb{R}^+$	$\mathrm{SO}(2)$	$\mathrm{SL}(2,\mathbb{Z})$
8	$\mathrm{SL}(3,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$	$\mathrm{SO}(3) \times \mathrm{SO}(2)$	$\mathrm{SL}(3,\mathbb{Z}) \times \mathrm{SL}(2,\mathbb{Z})$
7	$\mathrm{SL}(5,\mathbb{R})$	$\mathrm{SO}(5)$	$\mathrm{SL}(5,\mathbb{Z})$
6	$\mathrm{Spin}(5,5,\mathbb{R})$	$\big(\mathrm{Spin}(5) \times \mathrm{Spin}(5)\big)/\mathbb{Z}_2$	$\mathrm{Spin}(5,5,\mathbb{Z})$
5	$\mathrm{E}_6(\mathbb{R})$	$\mathrm{USp}(8)/\mathbb{Z}_2$	$\mathrm{E}_6(\mathbb{Z})$
4	$\mathrm{E}_7(\mathbb{R})$	$\mathrm{SU}(8)/\mathbb{Z}_2$	$\mathrm{E}_7(\mathbb{Z})$
3	$\mathrm{E}_8(\mathbb{R})$	$\mathrm{Spin}(16)/\mathbb{Z}_2$	$\mathrm{E}_8(\mathbb{Z})$

Amplitudes → **Action functional**

Toroidal compactifications yield the chain of **U-duality groups**

[Cremmer, Julia][Hull, Townsend]



D	G	K	$G(\mathbb{Z})$
10	$\mathrm{SL}(2,\mathbb{R})$	$\mathrm{SO}(2)$	$\mathrm{SL}(2,\mathbb{Z})$
9	$\mathrm{SL}(2,\mathbb{R}) \times \mathbb{R}^+$	$\mathrm{SO}(2)$	$\mathrm{SL}(2,\mathbb{Z})$
8	$\mathrm{SL}(3,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$	$\mathrm{SO}(3) \times \mathrm{SO}(2)$	$\mathrm{SL}(3,\mathbb{Z}) \times \mathrm{SL}(2,\mathbb{Z})$
7	$\mathrm{SL}(5,\mathbb{R})$	$\mathrm{SO}(5)$	$\mathrm{SL}(5,\mathbb{Z})$
6	$\mathrm{Spin}(5,5,\mathbb{R})$	$\left(\mathrm{Spin}(5) \times \mathrm{Spin}(5)\right)/\mathbb{Z}_2$	$\mathrm{Spin}(5,5,\mathbb{Z})$
5	$\mathrm{E}_6(\mathbb{R})$	$\mathrm{USp}(8)/\mathbb{Z}_2$	$\mathrm{E}_6(\mathbb{Z})$
4	$\mathrm{E}_7(\mathbb{R})$	$\mathrm{SU}(8)/\mathbb{Z}_2$	$\mathrm{E}_7(\mathbb{Z})$
3	$\mathrm{E}_8(\mathbb{R})$	$\mathrm{Spin}(16)/\mathbb{Z}_2$	$\mathrm{E}_8(\mathbb{Z})$

Amplitudes → **Action functional**

Einstein-Hilbert action

$$\int d^4x \sqrt{G} R$$

→

Einstein's equations

$$R_{\mu\nu} = 0$$

Higher-derivative action in type II string theory

Action functional:

$$\int d^{10-n}x\sqrt{G}\left[(\alpha')^3 f_0(g)\mathcal{R}^4 + (\alpha')^5 f_4(g)\partial^4\mathcal{R}^4 + \cdots\right]$$

Higher-derivative action in type II string theory

Action functional:

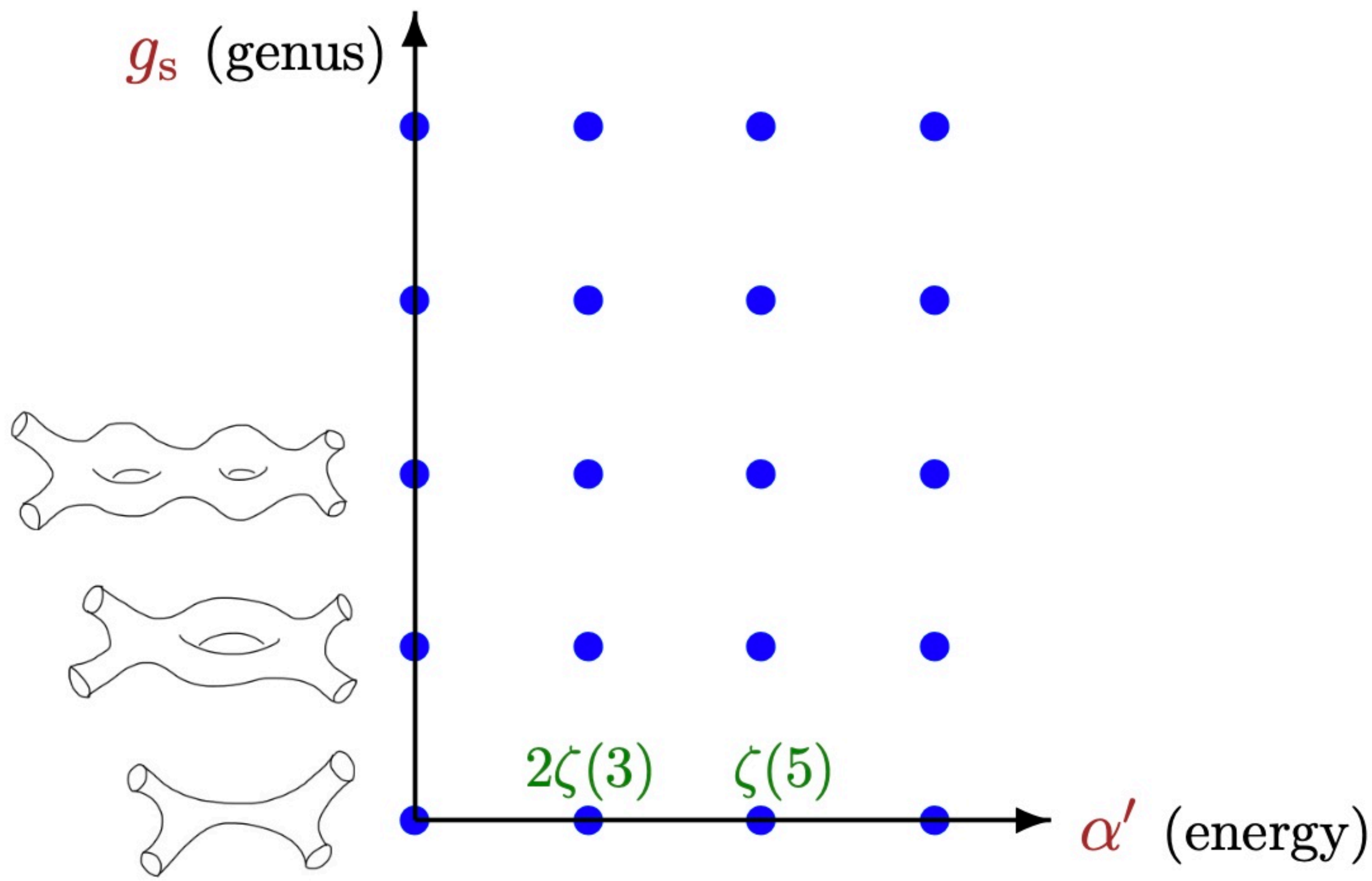
$$\int d^{10-n}x\sqrt{G}\left[(\alpha')^3 f_0(g)\mathcal{R}^4 + (\alpha')^5 f_4(g)\partial^4\mathcal{R}^4 + \cdots\right]$$

contraction of four Riemann tensors

Higher-derivative action in type II string theory

Action functional:

$$\int d^{10-n}x\sqrt{G}\left[(\alpha')^3 f_0(g)\mathcal{R}^4 + (\alpha')^5 f_4(g)\partial^4\mathcal{R}^4 + \cdots\right]$$

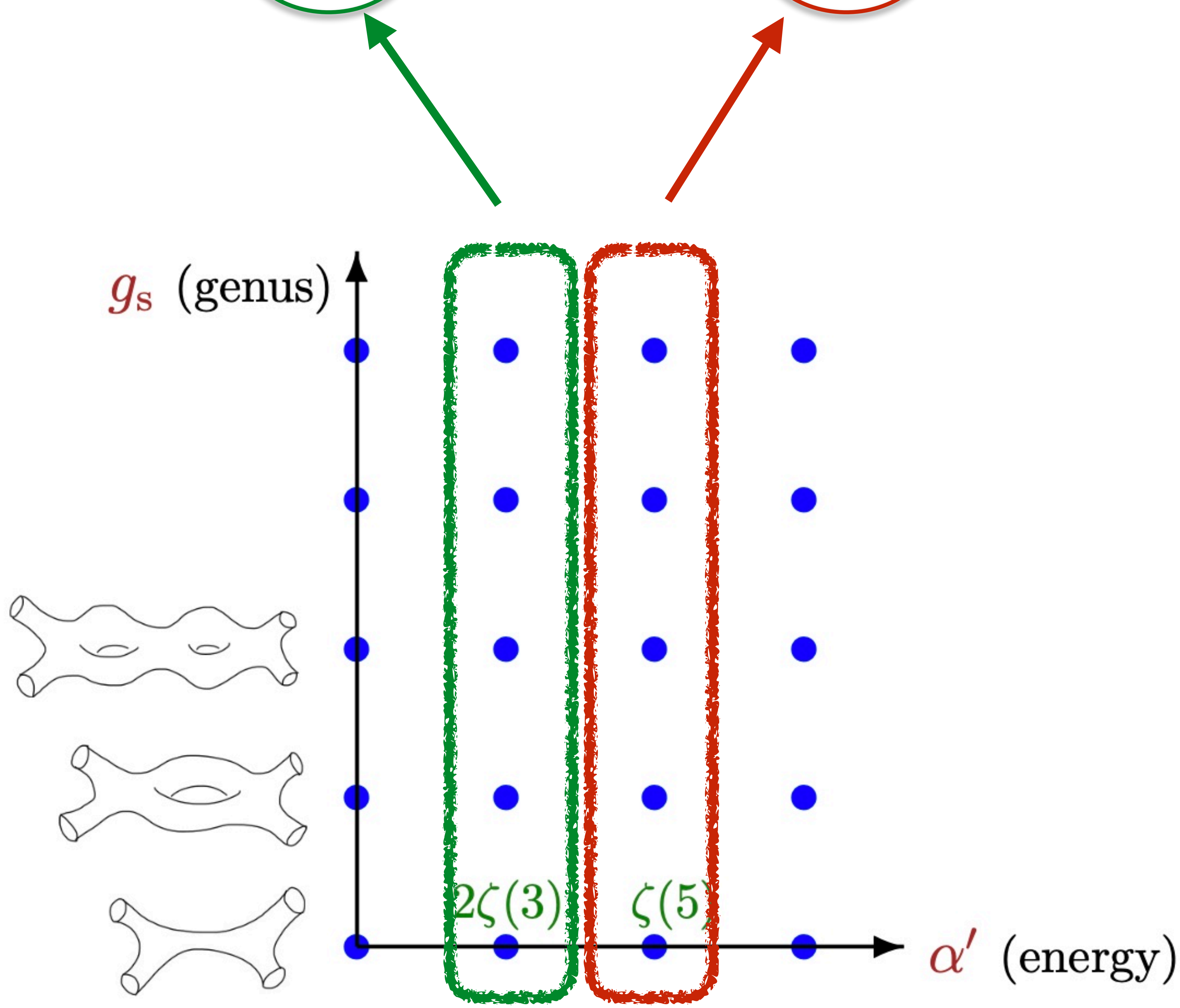


[picture stolen from Axel!]

Higher-derivative action in type II string theory

Action functional:

$$\int d^{10-n}x\sqrt{G}\left[(\alpha')^3 f_0(g)\mathcal{R}^4 + (\alpha')^5 f_4(g)\partial^4\mathcal{R}^4 + \cdots\right]$$



[picture stolen from Axel!]

Higher-derivative action in type II string theory

Action functional:

$$\int d^{10-n}x\sqrt{G}\left[(\alpha')^3 f_0(g)\mathcal{R}^4 + (\alpha')^5 f_4(g)\partial^4\mathcal{R}^4 + \cdots\right]$$

- → $f_0(g), f_4(g)$ are functions of $g \in E_{n+1}(\mathbb{R})/K$
- → must be **invariant** under U-duality $E_{n+1}(\mathbb{Z})$
- → supersymmetry requires that they are **Laplacian eigenfunctions**
- → well-defined **weak-coupling expansions** as $g_s \to 0$

Higher-derivative action in type II string theory

Action functional:

$$\int d^{10-n}x\sqrt{G}\left[(\alpha')^3 f_0(g)\mathcal{R}^4 + (\alpha')^5 f_4(g)\partial^4\mathcal{R}^4 + \cdots\right]$$

- $f_0(g), f_4(g)$ are functions of $g \in E_{n+1}(\mathbb{R})/K$
- must be **invariant** under U-duality $E_{n+1}(\mathbb{Z})$
- supersymmetry requires that they are **Laplacian eigenfunctions**
- well-defined **weak-coupling expansions** as $g_s \to 0$

defining properties of an **automorphic form!**

Example: type IIB in D=10

$$\int d^{10}x\sqrt{G}f_0(\tau)\mathcal{R}^4$$

Classical symmetry $G(\mathbb{R}) = SL(2,\mathbb{R})$

U-duality $G(\mathbb{Z}) = SL(2,\mathbb{Z})$

Moduli $\tau = x + iy \in SL(2,\mathbb{R})/SO(2)$

String coupling $g_s = y^{-1}$

Example: type IIB in D=10

$$\int d^{10}x\sqrt{G}f_0(\tau)\mathcal{R}^4$$

Classical symmetry $G = SL(2,\mathbb{R})$

U-duality $G(\mathbb{Z}) = SL(2,\mathbb{Z})$

$$f_0(\tau) = \overbrace{\underbrace{2\zeta(3)y^{3/2}}_{\text{tree-level}} + \underbrace{4\zeta(2)y^{-1/2}}_{\text{one-loop}}}^{\text{perturbative terms}} \overbrace{+2\pi\sum_{m\neq 0}\sqrt{|m|}\sigma_{-2}(m)e^{-S_{\text{inst}}(z)}\left[1+\mathcal{O}(y^{-1})\right]}^{\text{non-perturbative terms}}$$

amplitudes in the presence of instantons

Example: type IIB in D=10

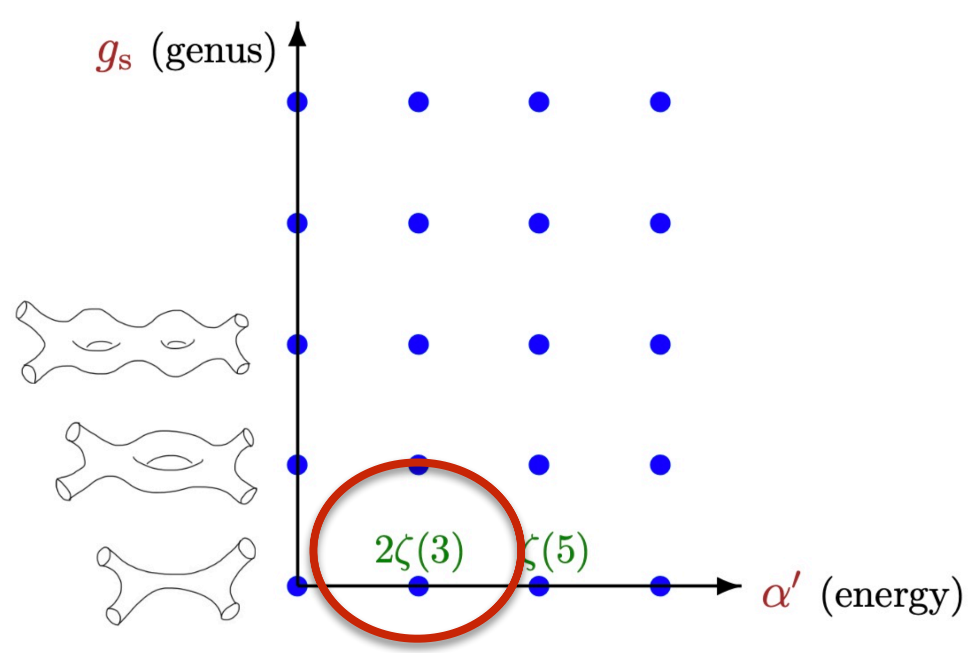
$$\int d^{10}x\sqrt{G}f_0(\tau)\mathcal{R}^4$$

Classical symmetry $G = SL(2,\mathbb{R})$

U-duality $G(\mathbb{Z}) = SL(2,\mathbb{Z})$

$$f_0(\tau) = \underbrace{\overbrace{2\zeta(3)y^{3/2}}_{\text{tree-level}} + \underbrace{4\zeta(2)y^{-1/2}}_{\text{one-loop}}}^{\text{perturbative terms}} \overbrace{+2\pi\sum_{m\neq 0}\sqrt{|m|}\sigma_{-2}(m)e^{-S_{\text{inst}}(z)}\left[1+\mathcal{O}(y^{-1})\right]}^{\text{non-perturbative terms}}$$

amplitudes in the presence of instantons



Example: type IIB in D=10

$$\int d^{10}x\sqrt{G}f_0(\tau)\mathcal{R}^4$$

Classical symmetry $G = SL(2,\mathbb{R})$

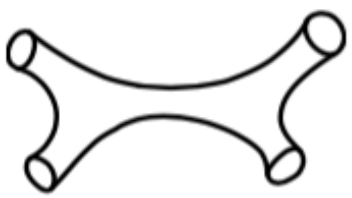
U-duality $G(\mathbb{Z}) = SL(2,\mathbb{Z})$

$$f_0(\tau) = \underbrace{\overbrace{2\zeta(3)y^{3/2}}^{}}_{\text{tree-level}} + \underbrace{4\zeta(2)y^{-1/2}}_{\text{one-loop}} + 2\pi\sum_{m\neq 0}\sqrt{|m|}\sigma_{-2}(m)e^{-S_{\text{inst}}(z)}\left[1+\mathcal{O}(y^{-1})\right]$$

perturbative terms: $2\zeta(3)y^{3/2} + 4\zeta(2)y^{-1/2}$ (tree-level, one-loop)

non-perturbative terms: $2\pi\sum_{m\neq 0}\sqrt{|m|}\sigma_{-2}(m)e^{-S_{\text{inst}}(z)}\left[1+\mathcal{O}(y^{-1})\right]$

tree-level



one-loop

amplitudes in the presence of instantons

instanton action

$$S_{\text{inst}}(z) := 2\pi\,|m|\,y - 2\pi i m x$$

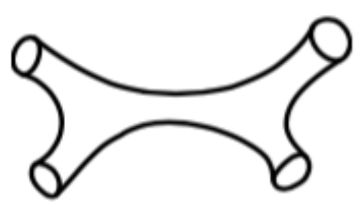
Example: type IIB in D=10

$$\int d^{10}x\sqrt{G}f_0(\tau)\mathcal{R}^4$$

Classical symmetry $G = SL(2,\mathbb{R})$

U-duality $G(\mathbb{Z}) = SL(2,\mathbb{Z})$

$$f_0(\tau) = \underbrace{\underbrace{2\zeta(3)y^{3/2}}_{\text{tree-level}} + \underbrace{4\zeta(2)y^{-1/2}}_{\text{one-loop}}}^{\text{perturbative terms}} \overbrace{+2\pi\sum_{m\neq 0}\sqrt{|m|}\sigma_{-2}(m)e^{-S_{\text{inst}}(z)}\left[1+\mathcal{O}(y^{-1})\right]}^{\text{non-perturbative terms}}$$



amplitudes in the presence of instantons

instanton action

$$S_{\text{inst}}(z) := 2\pi\,|m|\,y - 2\pi i m x$$

instanton measure

$$\sigma_{-2}(m) = \sum_{d|m} d^{-2}$$

Example: type IIB in D=10

$$\int d^{10}x\sqrt{G}\,f_0(\tau)\mathcal{R}^4$$

Classical symmetry $G = SL(2,\mathbb{R})$

U-duality $G(\mathbb{Z}) = SL(2,\mathbb{Z})$

$$f_0(\tau) = \overbrace{\underbrace{2\zeta(3)y^{3/2}}_{\text{tree-level}} + \underbrace{4\zeta(2)y^{-1/2}}_{\text{one-loop}}}^{\text{perturbative terms}} \overbrace{+2\pi\sum_{m\neq 0}\sqrt{|m|}\sigma_{-2}(m)e^{-S_{\text{inst}}(z)}\left[1+\mathcal{O}(y^{-1})\right]}^{\text{non-perturbative terms}}$$

amplitudes in the presence of instantons

Fourier expansion of an Eisenstein series at s=3/2 [Green, Gutperle]

$$f_0(\tau) = \sum_{(m,n)\neq(0,0)} \frac{y^{3/2}}{|m+n\tau|^3} \qquad \tau = x+iy$$

(See Sol Friedberg's lectures for the details)

Example: type IIB in D=10

$$\int d^{10}x\sqrt{G}f_0(\tau)\mathcal{R}^4$$

Classical symmetry $G = SL(2,\mathbb{R})$

U-duality $G(\mathbb{Z}) = SL(2,\mathbb{Z})$

$$f_0(\tau) = \overbrace{\underbrace{2\zeta(3)y^{3/2}}_{\text{tree-level}} + \underbrace{4\zeta(2)y^{-1/2}}_{\text{one-loop}}}^{\text{perturbative terms}} \overbrace{+2\pi\sum_{m\neq 0}\sqrt{|m|}\sigma_{-2}(m)e^{-S_{\text{inst}}(z)}\left[1+\mathcal{O}(y^{-1})\right]}^{\text{non-perturbative terms}}$$

amplitudes in the presence of instantons

Fourier coefficients of automorphic forms encode information about scattering amplitudes!

2. Automorphic forms and representation theory

Data:

- $G(\mathbb{R})$ real simple Lie group (e.g. $SL(n,\mathbb{R})$)
- $G(\mathbb{Z}) \subset G$ arithmetic subgroup (e.g. $SL(n,\mathbb{Z})$)

Definition:

An **automorphic form** is a smooth function $\varphi : G \longrightarrow \mathbb{C}$ satisfying

1. Automorphy: $\forall \gamma \in G(\mathbb{Z}), \;\; \varphi(\gamma g) = \varphi(g)$
2. φ is an eigenfunction of the ring of inv. diff. operators on G ($\mathcal{Z}$-finiteness)
3. φ has well-behaved growth conditions
4. K-finiteness

Data:

- $G(\mathbb{R})$ real simple Lie group (e.g. $SL(n,\mathbb{R})$)
- $G(\mathbb{Z}) \subset G$ arithmetic subgroup (e.g. $SL(n,\mathbb{Z})$)

Definition:

An **automorphic form** is a smooth function $\varphi : G \longrightarrow \mathbb{C}$ satisfying

1. Automorphy: $\forall \gamma \in G(\mathbb{Z}), \quad \varphi(\gamma g) = \varphi(g)$
2. φ is an eigenfunction of the ring of inv. diff. operators on G ($\mathcal{Z}$-finiteness)
3. φ has well-behaved growth conditions
4. K-finiteness

Focus on Eisenstein series

Eisenstein series

The **Langlands Eisenstein series** on a semi-simple Lie group is defined by:

$$E(\lambda, g) = \sum_{\gamma \in B(\mathbb{Z}) \backslash G(\mathbb{Z})} e^{\langle \lambda + \rho | H(\gamma g) \rangle}$$

Eisenstein series

The **Langlands Eisenstein series** on a semi-simple Lie group is defined by:

$$E(\lambda, g) = \sum_{\gamma \in B(\mathbb{Z})\backslash G(\mathbb{Z})} e^{\langle \lambda + \rho | H(\gamma g) \rangle}$$

Iwasawa decomposition: $G = BK = NAK$

$$A \sim \begin{pmatrix} * & & \\ & * & \\ & & * \end{pmatrix} \qquad N \sim \begin{pmatrix} 1 & * & * \\ & 1 & * \\ & & 1 \end{pmatrix}$$

Standard Borel subgroup (minimal parabolic) $B = NA$

Eisenstein series

The **Langlands Eisenstein series** on a semi-simple Lie group is defined by:

$$E(\lambda, g) = \sum_{\gamma \in B(\mathbb{Z})\backslash G(\mathbb{Z})} e^{\langle \lambda + \rho | H(\gamma g)\rangle}$$

Iwasawa decomposition: $G = BK = NAK$

Logarithm map: $H \ : \ G \to \mathfrak{h} = \mathrm{Lie}\, A \qquad H(nak) = \log a$

Eisenstein series

The **Langlands Eisenstein series** on a semi-simple Lie group is defined by:

$$E(\lambda, g) = \sum_{\gamma \in B(\mathbb{Z})\backslash G(\mathbb{Z})} e^{\langle \lambda + \rho | H(\gamma g) \rangle}$$

Iwasawa decomposition: $G = BK = NAK$

Logarithm map: $H \ : \ G \to \mathfrak{h} = \mathrm{Lie}\, A$ $\qquad H(nak) = \log a$

Weight: $\lambda \in \mathfrak{h}^\star \otimes \mathbb{C}$

Weyl vector: $\rho = \frac{1}{2} \sum_{\alpha > 0} \alpha$

Eisenstein series

The **Langlands Eisenstein series** on a semi-simple Lie group is defined by:

$$E(\lambda, g) = \sum_{\gamma \in B(\mathbb{Z}) \backslash G(\mathbb{Z})} e^{\langle \lambda + \rho | H(\gamma g) \rangle}$$

→ Converges absolutely on a subspace of $\mathfrak{h}^\star \otimes \mathbb{C}$

Godement's domain
$\{\lambda | \langle \lambda, \alpha \rangle > 1, \forall \alpha \in \Pi\}$

→ Can be continued to a meromorphic function on all of $\mathfrak{h}^\star \otimes \mathbb{C}$ [Langlands]

Eisenstein series

The **Langlands Eisenstein series** on a semi-simple Lie group is defined by:

$$E(\lambda, g) = \sum_{\gamma \in B(\mathbb{Z})\backslash G(\mathbb{Z})} e^{\langle \lambda + \rho | H(\gamma g) \rangle}$$

→ Converges absolutely on a subspace of $\mathfrak{h}^\star \otimes \mathbb{C}$

Godement's domain $\{\lambda | \langle \lambda, \alpha \rangle > 1, \forall \alpha \in \Pi\}$

→ Can be continued to a meromorphic function on all of $\mathfrak{h}^\star \otimes \mathbb{C}$ [Langlands]

→ Invariant: $E(\lambda, \gamma g k) = E(\lambda, g) \qquad \gamma \in G(\mathbb{Z}) \qquad k \in K$

→ Eigenfunction of the Laplacian: $\Delta_{G/K} E(\lambda, g) = \frac{1}{2}(\langle \lambda | \lambda \rangle - \langle \rho | \rho \rangle) E(\lambda, g)$

→ Functional relation: $E(\lambda, g) = M(w, \lambda) E(w\lambda, g), \quad \forall w \in W(\mathfrak{g})$

Automorphic representations

$$\mathcal{A}\left(G(\mathbb{Z})\backslash G(\mathbb{R})\right) = \{\text{space of automorphic forms on } G(\mathbb{R})\}$$

Automorphic representations

$$\mathcal{A}\left(G(\mathbb{Z})\backslash G(\mathbb{R})\right) = \{\text{space of automorphic forms on } G(\mathbb{R})\}$$

$$\cup$$

$$L^2(G(\mathbb{Z})\backslash G(\mathbb{R}))$$

Automorphic representations

$$\mathcal{A}\left(G(\mathbb{Z})\backslash G(\mathbb{R})\right) = \{\text{space of automorphic forms on } G(\mathbb{R})\}$$

Morally the group G acts on this space via the **right-regular representatio**

$$\left(\rho(h)\varphi\right)(g) = \varphi(gh) \qquad \text{for } \varphi \in \mathcal{A} \text{ and } h, g \in G$$

In reality, one must ensure compatibility with the K-action which leads to a so called Harish-Chandra module structure. But we skip that story here.

Automorphic representations

$$\mathcal{A}\left(G(\mathbb{Z})\backslash G(\mathbb{R})\right) = \{\text{space of automorphic forms on } G(\mathbb{R})\}$$

Morally the group G acts on this space via the **right-regular representatio**

$$(\rho(h)\varphi)\,(g) = \varphi(gh) \qquad\qquad \text{for } \varphi \in \mathcal{A} \text{ and } h, g \in G$$

In reality, one must ensure compatibility with the K-action which leads to a so called Harish-Chandra module structure. But we skip that story here.

Definition: *An **automorphic representation** π of G is an irreducible representation in the decomposition of $\mathcal{A}$ under the right-regular action (as a Harish-Chandra module).*

[Gelfand, Graev, Piatetski-Shapiro][Langlands]…

Toy model: Fourier analysis on $\mathbb{Z}\backslash\mathbb{R} \cong S^1$

Any function $f \in C^\infty(\mathbb{Z}\backslash\mathbb{R})$ can be decomposed into a Fourier series:

$$f(x) = \sum_{k\in\mathbb{Z}} c_k \psi_k(x)$$

$\psi_k \,:\, \mathbb{Z}\backslash\mathbb{R} \to U(1)$ $\qquad$ $\psi_k(x) = e^{2\pi i k x} \quad k \in \mathbb{Z},\, x \in \mathbb{R}$

Toy model: Fourier analysis on $\mathbb{Z}\backslash\mathbb{R} \cong S^1$

Any function $f \in C^\infty(\mathbb{Z}\backslash\mathbb{R})$ can be decomposed into a Fourier series:

$$f(x) = \sum_{k\in\mathbb{Z}} c_k \psi_k(x)$$

$$\psi_k \ : \ \mathbb{Z}\backslash\mathbb{R} \to U(1) \qquad\qquad \psi_k(x) = e^{2\pi i k x} \quad k \in \mathbb{Z},\, x \in \mathbb{R}$$

Moderate growth: restrict to **square integrable functions**

$$L^2(\mathbb{Z}\backslash\mathbb{R}) = \{ f \in C^\infty(\mathbb{Z}\backslash\mathbb{R}) \mid \sum_{k\in\infty} |c_k|^2 < \infty \}$$

$G = \mathbb{R}$ acts on $L^2(\mathbb{Z}\backslash\mathbb{R})$ via the regular representation

$$(\rho(y)f)(x) = f(x+y)$$

Toy model: Fourier analysis on $\mathbb{Z}\backslash\mathbb{R} \cong S^1$

Any function $f \in C^\infty(\mathbb{Z}\backslash\mathbb{R})$ can be decomposed into a Fourier series:

$$f(x) = \sum_{k\in\mathbb{Z}} c_k \psi_k(x)$$

$$\psi_k \,:\, \mathbb{Z}\backslash\mathbb{R} \to U(1) \qquad\qquad \psi_k(x) = e^{2\pi i k x} \quad k \in \mathbb{Z},\, x \in \mathbb{R}$$

Moderate growth: restrict to **square integrable functions**

$$L^2(\mathbb{Z}\backslash\mathbb{R}) = \{f \in C^\infty(\mathbb{Z}\backslash\mathbb{R}) \,|\, \sum_{k\in\infty} |c_k|^2 < \infty\}$$

$$L^2(\mathbb{Z}\backslash\mathbb{R}) = \bigoplus_{k\in\mathbb{Z}} \mathbb{C}\psi_k$$

Toy model: Fourier analysis on $\mathbb{Z}\backslash\mathbb{R} \cong S^1$

Any function $f \in C^\infty(\mathbb{Z}\backslash\mathbb{R})$ can be decomposed into a Fourier series:

$$f(x) = \sum_{k\in\mathbb{Z}} c_k \psi_k(x)$$

$$\psi_k : \mathbb{Z}\backslash\mathbb{R} \to U(1) \qquad \psi_k(x) = e^{2\pi i k x} \quad k \in \mathbb{Z},\, x \in \mathbb{R}$$

Moderate growth: restrict to **square integrable functions**

$$L^2(\mathbb{Z}\backslash\mathbb{R}) = \{f \in C^\infty(\mathbb{Z}\backslash\mathbb{R}) \mid \sum_{k\in\infty} |c_k|^2 < \infty\}$$

$$L^2(\mathbb{Z}\backslash\mathbb{R}) = \bigoplus_{k\in\mathbb{Z}} \mathbb{C}\psi_k$$

"automorphic representation"

Representation theory of Eisenstein series

Eisenstein series are attached to the (non-unitary) **principal series:**

$$I(\lambda) = \mathrm{Ind}_B^G \chi = \{ f : G \to \mathbb{C} \,|\, f(bg) = \chi(b) f(g), b \in B \}$$

Representation theory of Eisenstein series

Eisenstein series are attached to the (non-unitary) **principal series:**

$$I(\lambda) = \mathrm{Ind}_B^G \chi = \{f : G \to \mathbb{C} \,|\, f(bg) = \chi(b) f(g), b \in B\}$$

$$\chi = e^{\langle \lambda + \rho | H \rangle}$$

δ_B
in Sol's lectures

The theory of Eisenstein series then defines a map

$$E \; : \; I(\lambda) \to \mathcal{A}(G(\mathbb{Z})\backslash G(\mathbb{R}))$$

$$\chi \quad \longmapsto \quad \sum_{\gamma \in B(\mathbb{Z})\backslash G(\mathbb{Z})} e^{\langle \lambda + \rho | H(\gamma g) \rangle}$$

from the principal series to **the space of automorphic forms on** $G(\mathbb{R})$

3. Fourier coefficients

Whittaker coefficients

The periodicity $f(\tau+1) = f(\tau)$ generalises to

$$E(\lambda, ng) = E(\lambda, g) \qquad n \in N(\mathbb{Z})$$

Much more complicated since $N(\mathbb{Z})$ is **non-abelian**.

Whittaker coefficients

The periodicity $f(\tau+1) = f(\tau)$ generalises to

$$E(\lambda, ng) = E(\lambda, g) \qquad n \in N(\mathbb{Z})$$

Much more complicated since $N(\mathbb{Z})$ is **non-abelian**.

General structure:

$$E(\lambda, g) = E^{\text{const}}(\lambda, g) + \sum_{\psi} W_{\psi}(\lambda, g) + \cdots$$

constant term
(zero-mode)
perturbative effects

Whittaker coefficients

The periodicity $f(\tau+1) = f(\tau)$ generalises to

$$E(\lambda, ng) = E(\lambda, g) \qquad n \in N(\mathbb{Z})$$

Much more complicated since $N(\mathbb{Z})$ is **non-abelian**.

General structure:

$$E(\lambda, g) = E^{\text{const}}(\lambda, g) + \sum_{\psi} W_{\psi}(\lambda, g) + \cdots$$

↑

abelian coefficient
(non zero-modes)
non-perturbative effects

Whittaker coefficients

The periodicity $f(\tau+1) = f(\tau)$ generalises to

$$E(\lambda, ng) = E(\lambda, g) \qquad n \in N(\mathbb{Z})$$

Much more complicated since $N(\mathbb{Z})$ is **non-abelian**.

General structure:

$$E(\lambda, g) = E^{\mathrm{const}}(\lambda, g) + \sum_{\psi} W_{\psi}(\lambda, g) + \cdots$$

non-abelian coefficients
(non zero-modes)
non-perturbative effects

Whittaker coefficients

$$E(\lambda, g) = E^{\mathrm{const}}(\lambda, g) + \sum_{\psi} W_{\psi}(\lambda, g) + \cdots$$

Whittaker coefficients

$$E(\lambda, g) = E^{\text{const}}(\lambda, g) + \sum_{\psi} W_{\psi}(\lambda, g) + \cdots$$

$$W_{\psi}(g) = \int_{N(\mathbb{Z})\backslash N(\mathbb{R})} E(\lambda, ng)\overline{\psi(n)}dn$$

Whittaker coefficient

$$\psi \,:\, N(\mathbb{Z})\backslash N(\mathbb{R}) \to U(1)$$

unitary character on $N(\mathbb{R})$
trivial on $N(\mathbb{Z})$

Whittaker coefficients

$$E(\lambda, g) = E^{\text{const}}(\lambda, g) + \sum_{\psi} W_{\psi}(\lambda, g) + \cdots$$

$$W_{\psi}(g) = \int_{N(\mathbb{Z})\backslash N(\mathbb{R})} E(\lambda, ng)\overline{\psi(n)}dn$$

Whittaker coefficient

$$\psi \, : \, N(\mathbb{Z})\backslash N(\mathbb{R}) \to U(1)$$

unitary character on $N(\mathbb{R})$

→ $$\psi(n) = e^{2\pi i \sum_j m_j x_j}$$

(simple roots)

if all $m_j \neq 0$ then ψ is **generic**

if some $m_j = 0$ then ψ is **degenerate**

$m_j \in \mathbb{Z}$

$x_j \in \mathbb{R}$

Holomorphic modular form $f(\tau)$ $\tau \in \mathbb{H} \cong SL(2,\mathbb{R})/U(1)$

$$\psi\left(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}\right) = \psi(e^{xE_\alpha}) = e^{2\pi i m x}$$

$x \in \mathbb{R}$ $m \in \mathbb{Z}$

ψ generic $\longleftrightarrow$ $m \neq 0$

Holomorphic modular form $f(\tau)$ $\tau \in \mathbb{H} \cong SL(2,\mathbb{R})/U(1)$

$$\psi\left(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}\right) = \psi(e^{xE_\alpha}) = e^{2\pi i m x}$$

$x \in \mathbb{R}$ $m \in \mathbb{Z}$

ψ generic $\longleftrightarrow$ $m \neq 0$

$$W_m(\tau) = \int_0^1 f(\tau+1)e^{-2\pi i m u}\,du$$

$$= c(m)e^{2\pi i m \tau}$$

$$= c(m)q^m$$

Holomorphic modular form $f(\tau)$ $\tau \in \mathbb{H} \cong SL(2,\mathbb{R})/U(1)$

$$\psi\left(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}\right) = \psi(e^{xE_\alpha}) = e^{2\pi i m x}$$

$x \in \mathbb{R}$ $m \in \mathbb{Z}$

ψ generic $\longleftrightarrow$ $m \neq 0$

$$W_m(\tau) = \int_0^1 f(\tau+1) e^{-2\pi i m u}\, du$$

$$= c(m) e^{2\pi i m \tau}$$

$$= c(m) q^m$$

Non-holomorphic Eisenstein series

$$E(s,\tau) = \sum_{\substack{(m,n)\in\mathbb{Z}^2 \\ (m,n)=1}} \frac{y^s}{|m+n\tau|^{2s}}$$

$s \in \mathbb{C}$

$\tau = x + iy \in \mathbb{H}$

Holomorphic modular form $f(\tau)$ $\tau \in \mathbb{H} \cong SL(2,\mathbb{R})/U(1)$

$$\psi\left(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}\right) = \psi(e^{xE_\alpha}) = e^{2\pi i m x}$$

$$x \in \mathbb{R} \qquad m \in \mathbb{Z}$$

ψ generic $\longleftrightarrow$ $m \neq 0$

$$W_m(\tau) = \int_0^1 f(\tau+1)e^{-2\pi i m u}\, du$$

$$= c(m)e^{2\pi i m \tau}$$

$$= c(m)q^m$$

Non-holomorphic Eisenstein series

$$E(s,\tau) = \sum_{\substack{(m,n)\in\mathbb{Z}^2 \\ (m,n)=1}} \frac{y^s}{|m+n\tau|^{2s}}$$

$$s \in \mathbb{C}$$

$$\tau = x + iy \in \mathbb{H}$$

$$W_m(\tau) = \int_0^1 E(s,\tau+u)e^{-2\pi i m u} du = \frac{\sqrt{y}}{\xi(2s)}\sigma_{1-2s}(m)K_{s-1/2}(2\pi|m|y)e^{2\pi i m x}$$

$$\sigma_{1-2s}(m) = \sum_{d|m} d^{1-2s}$$

(modified) Bessel function

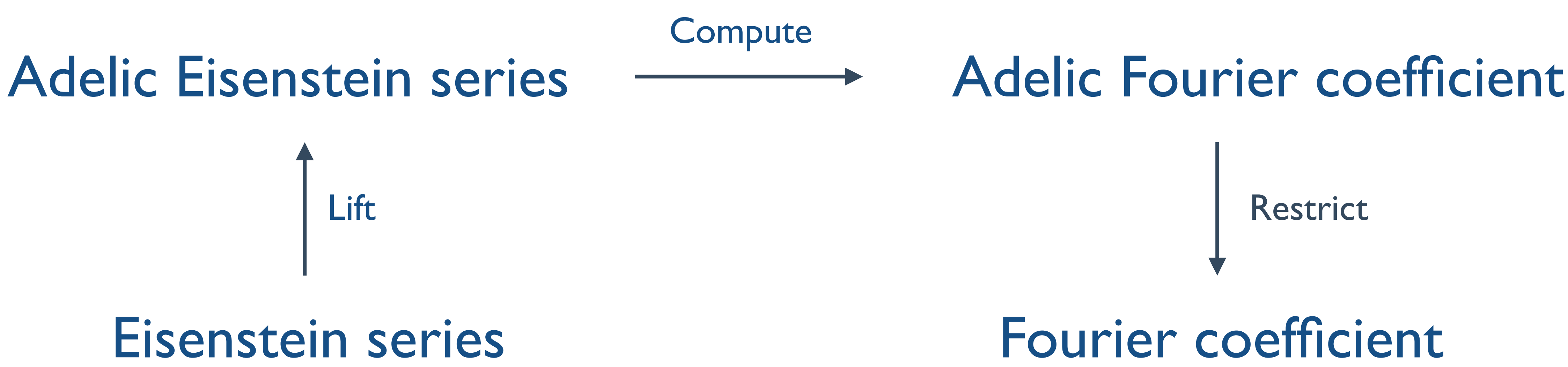
Adelic framework

> *An efficient, but abstract, way to approach the subject of automorphic forms is by the introduction of adeles, rather ungainly objects that nevertheless, once familiar, spare much unnecessary thought and many useless calculations.*

— Robert P. Langlands

Adelic framework

> *An efficient, but abstract, way to approach the subject of automorphic forms is by the introduction of adeles, rather ungainly objects that nevertheless, once familiar, spare much unnecessary thought and many useless calculations.*
>
> — Robert P. Langlands



Adelic framework

arithmetic groups	$G(\mathbb{Z}) \subset G(\mathbb{R})$	$\longrightarrow$	$G(\mathbb{Q}) \subset G(\mathbb{A})$
space of automorphic forms	$\mathcal{A}\big(G(\mathbb{Z})\backslash G(\mathbb{R})\big)$	$\longrightarrow$	$\mathcal{A}\big(G(\mathbb{Q})\backslash G(\mathbb{A})\big)$

Adelic framework

arithmetic groups	$G(\mathbb{Z}) \subset G(\mathbb{R})$	$\longrightarrow$	$G(\mathbb{Q}) \subset G(\mathbb{A})$
space of automorphic forms	$\mathcal{A}\big(G(\mathbb{Z})\backslash G(\mathbb{R})\big)$	$\longrightarrow$	$\mathcal{A}\big(G(\mathbb{Q})\backslash G(\mathbb{A})\big)$
	$\bigcup$		$\bigcup$
Eisenstein series	$\sum_{\gamma \in B(\mathbb{Z})\backslash G(\mathbb{Z})} e^{\langle \lambda+\rho \| H(\gamma g)\rangle}$	$\longrightarrow$	$\sum_{\gamma \in B(\mathbb{Q})\backslash G(\mathbb{Q})} e^{\langle \lambda+\rho \| H(\gamma g)\rangle}$

$$\lambda \in \mathfrak{h}^\star \otimes \mathbb{C} \qquad\qquad H : G \to \mathfrak{h}$$

Theorem [Jacquet, Langlands]**:** *The generic Whittaker coefficient is Eulerian*

$$W_\psi(\lambda, g) = \int_{N(\mathbb{A})} \chi(w_0 n g)\, \overline{\psi(n)}\, dn = \prod_p W_{\psi_p}$$

Theorem [Jacquet, Langlands]**:** *The generic Whittaker coefficient is Eulerian*

$$W_\psi(\lambda, g) = \int_{N(\mathbb{A})} \chi(w_0 n g)\, \overline{\psi(n)}\, dn = \prod_p W_{\psi_p}$$

$w_0 =$ longest element of $W(\mathfrak{g})$

Compare: **Euler product** of Riemann zeta function

$$\zeta(s) = \prod_{p<\infty} \frac{1}{1-p^{-s}}$$

Theorem [Jacquet, Langlands]**:** *The generic Whittaker coefficient is Eulerian*

$$W_\psi(\lambda, g) = \int_{N(\mathbb{A})} \chi(w_0 n g)\, \overline{\psi(n)}\, dn = \prod_p W_{\psi_p}$$

$$W_{\psi_\infty} = \int_{N(\mathbb{R})} \chi_\infty(w_0 n a_\infty) \overline{\psi_\infty(n)}\, dn$$

$$W_{\psi_p} = \int_{N(\mathbb{Q}_p)} \chi_p(w_0 n a_p) \overline{\psi_p(n)}\, dn$$

General Fourier coefficients

- $P = LU$ **standard parabolic** of G

Example:

$$P = \left\{ \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ & & * & * \\ & & & * \end{pmatrix} \right\} = LU$$

General Fourier coefficients

- $P = LU$ **standard parabolic** of G
- **unitary character** $\psi_U : U(\mathbb{Q})\backslash U(\mathbb{A}) \to U(1)$

General Fourier coefficients

- $P = LU$ **standard parabolic** of G
- **unitary character** $\psi_U : U(\mathbb{Q})\backslash U(\mathbb{A}) \to U(1)$
- For any automorphic form φ we have the U ***-coefficient***

$$F_{\psi_U}(g) = \int_{U(\mathbb{Q})\backslash U(\mathbb{A})} \varphi(ug)\overline{\psi_U(u)}du$$

Also known as "**unipotent period integral**".

$$F_{\psi_U}(g) = \int_{U(\mathbb{Q})\backslash U(\mathbb{A})} \varphi(ug)\overline{\psi_U(u)}du$$

- These are **not Eulerian** in general
- $F_{\psi_U}(ug) = \psi_U(u)F_{\psi_U}(g) \qquad \forall u \in U$
- Very difficult to compute in general
- Idea: consider **special types** of automorphic representations (motivated by string theory)

Perturbative limit - choices of unipotent subgroups

$$P = LU$$

Levi subgroups:

Decompactification limit

$L = E_7$

- perturbative parameter: radius of decompactified circle
- non-perturbative effects: KK-instantons, BPS-instantons

String perturbation limit

$L = D_7$

- perturbative parameter: string coupling
- non-perturbative effects: D-instantons, NS5-instantons

M-theory limit

$L = A_7$

- perturbative parameter: volume of M-theory torus
- non-perturbative effects: M2- & M5-instantons

Example: $G = SO(5,5)$ type II string theory on T^4 [Green, Russo, Vanhove]

Higher-derivative coupling: $$\int d^4x\sqrt{G}f_0(g)\mathcal{R}^4$$

Example: $G = SO(5,5)$ type II string theory on T^4 [Green, Russo, Vanhove]

Higher-derivative coupling: $\int d^4x\sqrt{G} f_0(g)\mathcal{R}^4$

Eisenstein series: $f_0(g) = E(2s\Lambda_1 - \rho, g)$ $s = 3/2$

Consider string theory limit: $P = LU$

$L = SO(4,4)$

Example: $G = SO(5,5)$ type II string theory on T^4 [Green, Russo, Vanhove]

Higher-derivative coupling: $$\int d^4x \sqrt{G} f_0(g) \mathcal{R}^4$$

Eisenstein series: $f_0(g) = E(2s\Lambda_1 - \rho, g)$ $\quad s = 3/2$

Consider string theory limit: $P = LU$

$$L = SO(4,4)$$

Perturbative contribution: Constant term

$$\int_{U(\mathbb{Z})\backslash U(\mathbb{R})} E(3\Lambda_1 - \rho, ug) du = \frac{2\zeta(3)}{g_s^2} + E^{SO(4,4)}$$

Example: $G = SO(5,5)$ type II string theory on T^4 [Green, Russo, Vanhove]

Higher-derivative coupling: $$\int d^4x\sqrt{G}f_0(g)\mathcal{R}^4$$

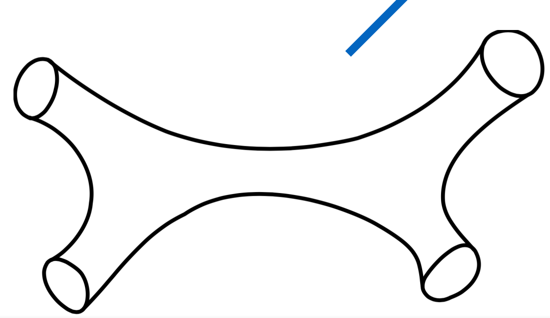
Eisenstein series: $f_0(g) = E(2s\Lambda_1 - \rho, g) \qquad s = 3/2$

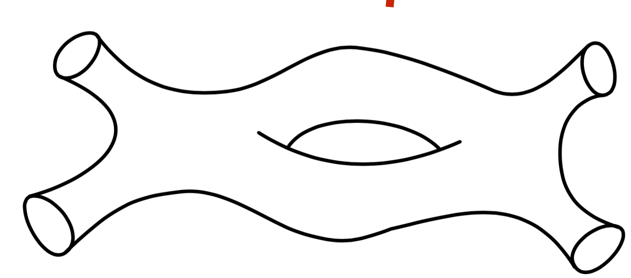
Consider string theory limit: $P = LU$

$$L = SO(4,4)$$

Perturbative contribution: Constant term

$$\int_{U(\mathbb{Z})\backslash U(\mathbb{R})} E(3\Lambda_1 - \rho, ug)du = \frac{2\zeta(3)}{g_s^2} + E^{SO(4,4)}$$





4. Small representations

Minimal automorphic representations

Definition: *An automorphic representation*

$$\pi = \bigotimes_{p \leq \infty} \pi_p$$

is minimal if each factor π_p *has smallest non-trivial Gelfand-Kirillov (functional) dimension.*

[Joseph][Brylinski, Kostant][Ginzburg, Rallis, Soudry][Kazhdan, Savin]....

Minimal automorphic representations

Definition: *An automorphic representation*

$$\pi = \bigotimes_{p\leq\infty} \pi_p$$

is minimal if each factor π_p *has smallest non-trivial Gelfand-Kirillov (functional) dimension.*

[Joseph][Brylinski, Kostant][Ginzburg, Rallis, Soudry][Kazhdan, Savin]....

Automorphic forms $\varphi \in \pi_{min}$ are characterised by having **very few non-vanishing Fourier coefficients**.

[Ginzburg, Rallis, Soudry]

Physical couplings

$g \in E_n(\mathbb{R})$

$$\int d^{11-n}x\sqrt{G}f_0(g)\mathcal{R}^4 \qquad f_0(g) = E(3/2, g) \qquad s = 3/2$$

$$\int d^{11-n}\sqrt{G}f_4(g)\partial^4\mathcal{R}^4 \qquad f_4(g) = E(5/2, g) \qquad s = 5/2$$

These partition functions are Eisenstein series attached to **small automorphic representations** of G.

[Green, Miller, Vanhove][Pioline]

minimal automorphic representation	**next-to-minimal automorphic representation**
π_{min}	π_{ntm}

1/2 - BPS

1/4 - BPS

Example for min rep of $SL(5,\mathbb{A})$

$$P = LU \qquad L = SL(4)\times GL(1) \qquad U = \left\{\begin{pmatrix} 1 & & & & * \\ & 1 & & & * \\ & & 1 & & * \\ & & & 1 & * \\ & & & & 1 \end{pmatrix}\right\}$$

Example for min rep of $SL(5,\mathbb{A})$

$$P = LU \qquad L = SL(4)\times GL(1) \qquad U = \left\{\begin{pmatrix} 1 & & & & * \\ & 1 & & & * \\ & & 1 & & * \\ & & & 1 & * \\ & & & & 1 \end{pmatrix}\right\}$$

$$\psi_U\left(\begin{pmatrix} 1 & & & & * \\ & 1 & & & * \\ & & 1 & & * \\ & & & 1 & x \\ & & & & 1 \end{pmatrix}\right) = e^{2\pi i k x} \qquad x\in\mathbb{A},\ k\in\mathbb{Q} \qquad \mathrm{rank}(\psi_U) = 1$$

Example for min rep of $SL(5,\mathbb{A})$

$$P = LU \qquad L = SL(4)\times GL(1) \qquad U = \left\{\begin{pmatrix} 1 & & & & * \\ & 1 & & & * \\ & & 1 & & * \\ & & & 1 & * \\ & & & & 1 \end{pmatrix}\right\}$$

$$\psi_U\left(\begin{pmatrix} 1 & & & & * \\ & 1 & & & * \\ & & 1 & & * \\ & & & 1 & x \\ & & & & 1 \end{pmatrix}\right) = e^{2\pi i k x} \qquad x\in\mathbb{A},\ k\in\mathbb{Q} \qquad \mathrm{rank}(\psi_U) = 1$$

$$F_{\psi_U}(1) = \frac{2}{\xi(2s)}\sigma_{2s-4}(k)|k|^{2-s}K_{s-2}(2\pi|k|)$$

Example for min rep of $SL(5,\mathbb{A})$

$$P = LU \qquad L = SL(4)\times GL(1) \qquad U = \left\{\begin{pmatrix} 1 & & & & * \\ & 1 & & & * \\ & & 1 & & * \\ & & & 1 & * \\ & & & & 1 \end{pmatrix}\right\}$$

$$\psi_U\left(\begin{pmatrix} 1 & & & & * \\ & 1 & & & * \\ & & 1 & & * \\ & & & 1 & x \\ & & & & 1 \end{pmatrix}\right) = e^{2\pi i k x} \qquad x\in\mathbb{A},\ k\in\mathbb{Q} \qquad \mathrm{rank}(\psi_U) = 1$$

$$F_{\psi_U}(1) = \frac{2}{\xi(2s)}\sigma_{2s-4}(k)|k|^{2-s}K_{s-2}(2\pi|k|)$$

For $s=3/2$ this captures the contributions from **M2-brane instantons** in M-theory compactified on T^4 [Green, Miller, Vanhove]

Instanton measure: $\sigma_{-1}(k) = \sum_{d|k} d^{-1}$

Theorem: w/ [Gustafsson, Gourevitch, Kleinschmidt, Sahi]

Let G be a ***semisimple, simply laced Lie group****.*

Then all Fourier coefficients of $\varphi \in \pi_{ntm}$ are completely determined by degenerate Whittaker vectors of the form

$$W_{\psi_\alpha}(\varphi, g) = \int_{N(\mathbb{Q})\backslash N(\mathbb{A})} \varphi(ng)\overline{\psi_\alpha(n)}dn$$

$$W_{\psi_{\alpha,\beta}}(\varphi, g) = \int_{N(\mathbb{Q})\backslash N(\mathbb{A})} \varphi(ng)\overline{\psi_{\alpha,\beta}(n)}dn$$

where (α, β) are commuting simple roots.

This generalises earlier results of [Ginzburg, Rallis, Soudry][Miller, Sahi]

Theorem: w/ **[Gustafsson, Gourevitch, Kleinschmidt, Sahi]**

Let G be a ***semisimple, simply laced Lie group****.*

Then all Fourier coefficients of $\varphi \in \pi_{ntm}$ are completely determined by degenerate Whittaker vectors of the form

$$W_{\psi_\alpha}(\varphi, g) = \int_{N(\mathbb{Q})\backslash N(\mathbb{A})} \varphi(ng)\overline{\psi_\alpha(n)}dn$$

$$W_{\psi_{\alpha,\beta}}(\varphi, g) = \int_{N(\mathbb{Q})\backslash N(\mathbb{A})} \varphi(ng)\overline{\psi_{\alpha,\beta}(n)}dn$$

where (α, β) are commuting simple roots.

This allows to extract instanton effects to 1/4-BPS couplings

Corollary: [Gourevitch, Gustafsson, Kleinschmidt, Sahi, DP]

Any $\varphi \in \pi_{ntm}$ *can be expanded as*

$$\varphi(g) = \sum_{\mathcal{O} \in WF(\pi_{ntm})} \mathcal{F}_{\mathcal{O}}(g)$$

where each $\mathcal{F}_{\mathcal{O}}$ *is linearly determined by degenerate Whittaker coefficients*

Corollary: [Gourevitch, Gustafsson, Kleinschmidt, Sahi, DP]

Any $\varphi \in \pi_{ntm}$ *can be expanded as*

$$\varphi(g) = \sum_{\mathcal{O} \in WF(\pi_{ntm})} \mathcal{F}_{\mathcal{O}}(g)$$

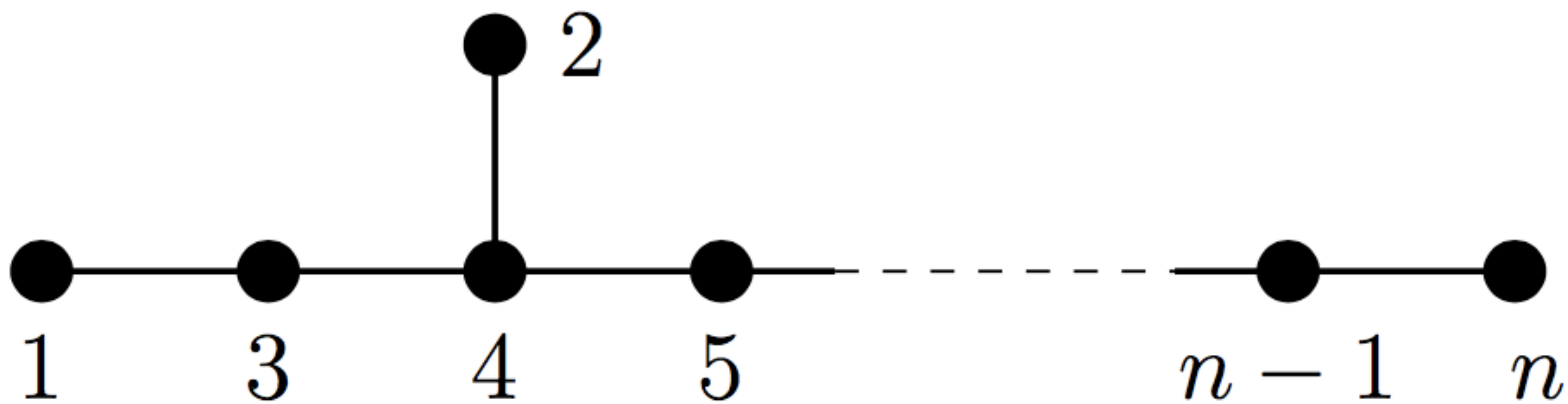
where each $\mathcal{F}_{\mathcal{O}}$ *is linearly determined by degenerate Whittaker coefficients*

Recall orbit coefficients in Sol's lecture!

- $\pi_{min} \, : \, \varphi(g) = \mathcal{F}_{\mathcal{O}_0}(g) + \mathcal{F}_{\mathcal{O}_{min}}(g)$

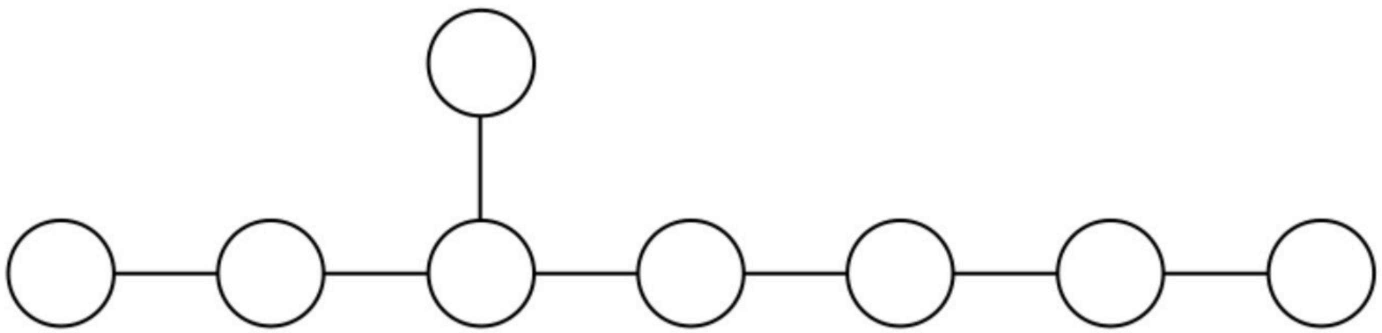
- $\pi_{ntm} \, : \, \varphi(g) = \mathcal{F}_{\mathcal{O}_0}(g) + \mathcal{F}_{\mathcal{O}_{min}}(g) + \mathcal{F}_{\mathcal{O}_{ntm}}(g)$

Focus on exceptional groups



Functional dimension of minimal representations:

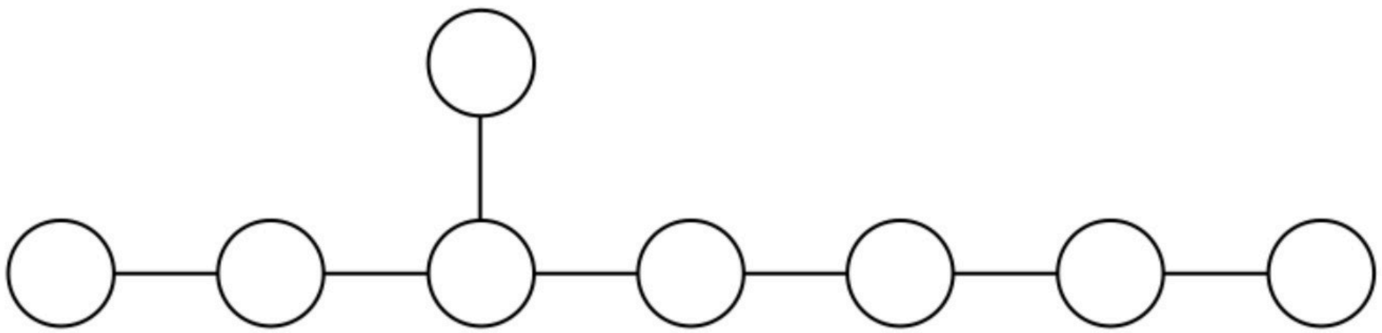
$$\mathrm{GKdim}\,\pi_{min} = \begin{cases} 11, & E_6 \\ 17, & E_7 \\ 29, & E_8 \end{cases}$$



$$\mathrm{GKdim}(\pi_{min}) = 29$$

(dim of generic rep: 120)

$$E_8 = \mathbf{1} \oplus \mathbf{56} \oplus (E_7 \oplus \mathbf{1}) \oplus \mathbf{56} \oplus \mathbf{1}$$



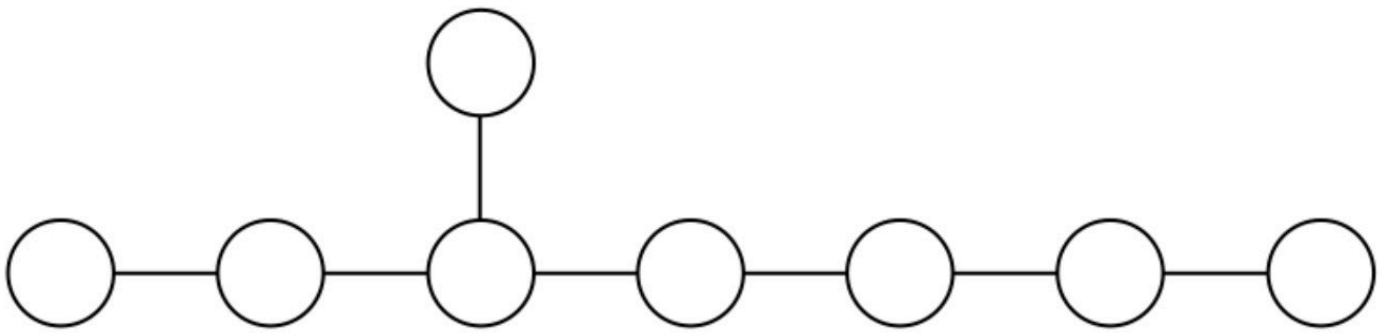
$$\mathrm{GKdim}(\pi_{min}) = 29$$

(dim of generic rep: 120)

$$E_8 = \mathbf{1} \oplus \mathbf{56} \oplus (E_7 \oplus \mathbf{1}) \oplus \mathbf{56} \oplus \mathbf{1}$$

$$\bigcup$$

$$\mathbf{28} \oplus \mathbf{1}$$



$$\mathrm{GKdim}(\pi_{min}) = 29$$

(dim of generic rep: 120)

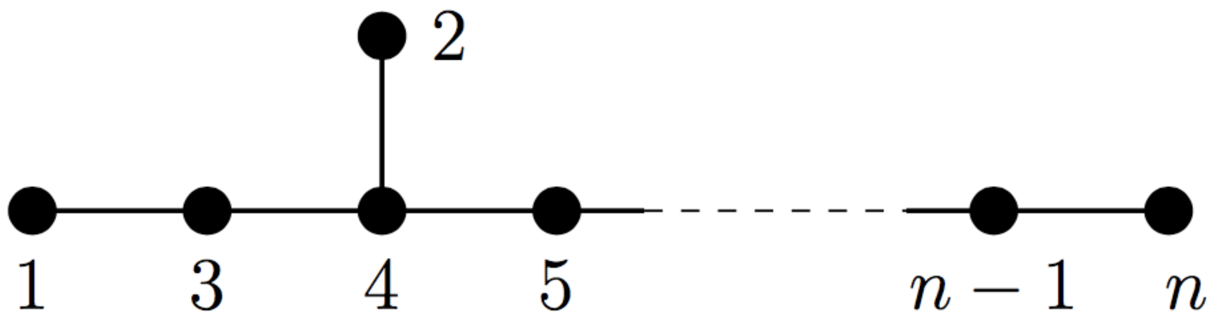
$$E_8 = \mathbf{1} \oplus \mathbf{56} \oplus (E_7 \oplus \mathbf{1}) \oplus \mathbf{56} \oplus \mathbf{1}$$

$$\cup$$

$$\mathbf{28} \oplus \mathbf{1}$$

Minimal representation realized by the group action on functions of these 29 variables

Minimal representation of exceptional groups



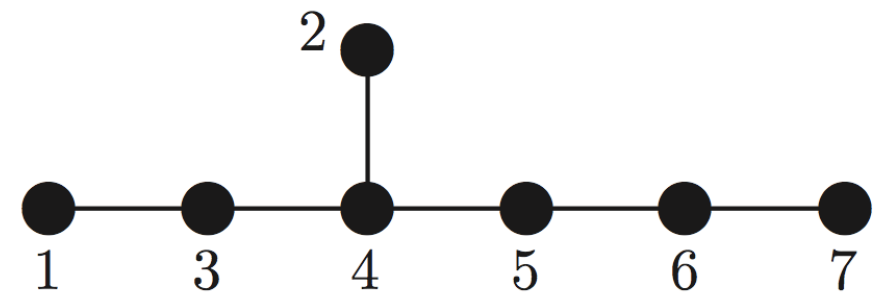
$$\mathrm{GKdim}\,\pi_{min} = \begin{cases} 11, & E_6 \\ 17, & E_7 \\ 29, & E_8 \end{cases}$$

Minimal automorphic forms can be obtained as **special values of Eisenstein series**

[Ginzburg, Rallis, Soudry]
(also [Green, Miller, Vanhove])

Theorem: $E(2s\Lambda_1 - \rho, g) \in \pi_{min} \quad \text{when} \quad s = 3/2$

Example: $G = E_7$

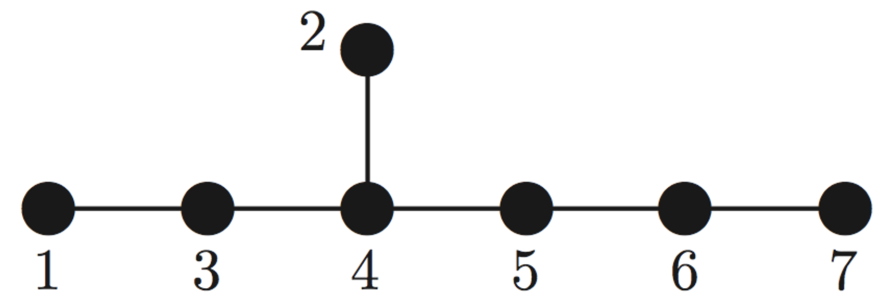


$$E(2s\Lambda_1 - \rho, g) = \varphi(g) \qquad E_7 = \mathbf{27} \oplus (E_6 \oplus \mathbf{1}) \oplus \mathbf{27}$$

Theorem: $$\pi_{min} \ni \varphi(g) = \varphi_U + \sum_{\gamma \in \mathrm{Stab}_L(\psi_{\alpha_7}) \backslash L(\mathbb{Q})} F_{\psi_{\alpha_7}}(\gamma g)$$

[Ginzburg, Rallis, Soudry]

Example: $G = E_7$



$$E(2s\Lambda_1 - \rho, g) = \varphi(g) \qquad E_7 = \mathbf{27} \oplus (E_6 \oplus \mathbf{1}) \oplus \mathbf{27}$$

Theorem: $$\pi_{min} \ni \varphi(g) = \varphi_U + \sum_{\gamma \in \mathrm{Stab}_L(\psi_{\alpha_7})\backslash L(\mathbb{Q})} F_{\psi_{\alpha_7}}(\gamma g)$$

[Ginzburg, Rallis, Soudry]

The complete expansion is given in terms of the **smallest non-trivial** character variety orbit

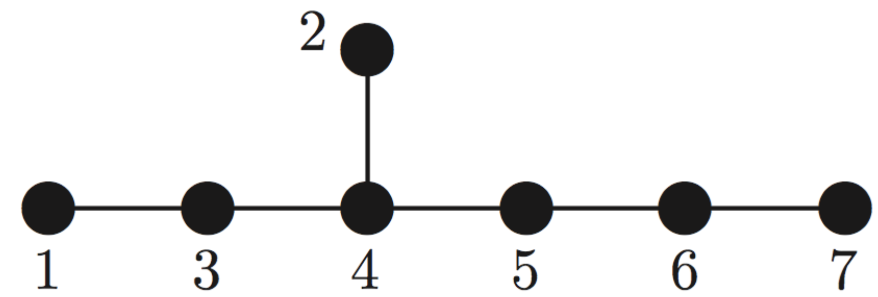
$$\mathrm{Stab}_L(\psi_{\alpha_7})\backslash L \cong (SO(5,5) \times \mathbb{Q}^{16})\backslash E_6(\mathbb{Q})$$

$$\dim = 17$$

[Miller, Sahi]

dimension of the minimal representation!

Example: $G = E_7$



$$E(2s\Lambda_1 - \rho, g) = \varphi(g) \qquad E_7 = \mathbf{27} \oplus (E_6 \oplus \mathbf{1}) \oplus \mathbf{27}$$

Theorem: $$\pi_{min} \ni \varphi(g) = \varphi_U + \sum_{\gamma \in \mathrm{Stab}_L(\psi_{\alpha_7}) \backslash L(\mathbb{Q})} F_{\psi_{\alpha_7}}(\gamma g)$$

[Ginzburg, Rallis, Soudry]

The complete expansion is given in terms of the **smallest non-trivial** character variety orbit

$$\mathrm{Stab}_L(\psi_{\alpha_7}) \backslash L \cong (SO(5,5) \times \mathbb{Q}^{16}) \backslash E_6(\mathbb{Q})$$

$$\dim = 17$$

[Miller, Sahi]

dimension of the minimal representation!

Physically, the sum is over 1/2 BPS instanton charges in string theory on T^6!

Theorem: w/ **[Gustafsson, Gourevitch, Kleinschmidt, Sahi]**

Consider $G = E_8$ *. Any automorphic form* $\eta \in \pi_{ntm}$ *may then be expanded as follows*

$$\eta(g) = \mathcal{F}_{S_\alpha,0}[\eta](g) + \sum_{\tilde\gamma\in\Gamma_{\varphi_0}} \Big(\mathcal{W}_{\varphi_0}[\eta](\tilde\gamma g) + \sum_{i\in I}\sum_{\gamma\in\Gamma_i}\sum_{\psi\in\mathfrak{g}^\times_{-\alpha_i}} \mathcal{W}_{\varphi_0+\psi}[\eta](\gamma\tilde\gamma g)\Big)$$

$$+ \sum_{\tilde\gamma\in\Lambda}\int_{V(\mathbb{A})} \mathcal{W}_{\varphi_0+\psi_0}[\eta](v\tilde\gamma g)\, dv + \sum_{w\in W(\mathbb{K})} \Big(\sum_{c\in\mathbb{K}^\times} \mathcal{W}_{c\varphi_0}[\eta](wsg) +$$

$$\sum_{c\in\mathbb{K}^\times}\sum_{i\in I}\sum_{\gamma\in\Gamma_i}\sum_{\psi\in\mathfrak{g}^\times_{-\alpha_i}} \mathcal{W}_{c\varphi_0+\psi}[\eta](\gamma wsg) + \sum_{\tilde\gamma\in\mathcal{M}}\int_{V(\mathbb{A})} \mathcal{W}_{\varphi_0+\psi_0}[\eta](v\tilde\gamma wsg)\Big)$$

Theorem: w/ **[Gustafsson, Gourevitch, Kleinschmidt, Sahi]**

Consider $G = E_8$. *Any automorphic form* $\eta \in \pi_{ntm}$ *may then be expanded as follows*

$$\eta(g) = \mathcal{F}_{S_\alpha,0}[\eta](g) + \sum_{\tilde\gamma \in \Gamma_{\varphi_0}} \Big(\mathcal{W}_{\varphi_0}[\eta](\tilde\gamma g) + \sum_{i \in I} \sum_{\gamma \in \Gamma_i} \sum_{\psi \in \mathfrak{g}^\times_{-\alpha_i}} \mathcal{W}_{\varphi_0+\psi}[\eta](\gamma\tilde\gamma g) \Big)$$

$$+ \sum_{\tilde\gamma \in \Lambda} \int_{V(\mathbb{A})} \mathcal{W}_{\varphi_0+\psi_0}[\eta](v\tilde\gamma g)\, dv + \sum_{w \in W(\mathbb{K})} \Big(\sum_{c \in \mathbb{K}^\times} \mathcal{W}_{c\varphi_0}[\eta](wsg) +$$

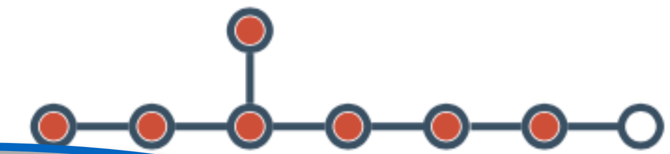
$$\sum_{c \in \mathbb{K}^\times} \sum_{i \in I} \sum_{\gamma \in \Gamma_i} \sum_{\psi \in \mathfrak{g}^\times_{-\alpha_i}} \mathcal{W}_{c\varphi_0+\psi}[\eta](\gamma wsg) + \sum_{\tilde\gamma \in \mathcal{M}} \int_{V(\mathbb{A})} \mathcal{W}_{\varphi_0+\psi_0}[\eta](v\tilde\gamma wsg) \Big)$$

contribution from $\mathcal{O}_{min}$ to the abelian term

$$\int_{Z(F)\backslash Z(\mathbb{A})} \eta(ug)du \qquad\qquad Z = [U_{Heis}, U_{Heis}]$$

Theorem: w/ **[Gustafsson, Gourevitch, Kleinschmidt, Sahi]**

Consider $G = E_8$. *Any automorphic form* $\eta \in \pi_{ntm}$ *may then be expanded as follows*



$$\eta(g) = \mathcal{F}_{S_\alpha,0}[\eta](g) + \sum_{\tilde\gamma\in\Gamma_{\varphi_0}} \Big(\mathcal{W}_{\varphi_0}[\eta](\tilde\gamma g) + \sum_{i\in I}\sum_{\gamma\in\Gamma_i}\sum_{\psi\in\mathfrak{g}^\times_{-\alpha_i}} \mathcal{W}_{\varphi_0+\psi}[\eta](\gamma\tilde\gamma g)\Big)$$
$$+ \sum_{\tilde\gamma\in\Lambda_{V(\mathbb{A})}} \int \mathcal{W}_{\varphi_0+\psi_0}[\eta](v\tilde\gamma g)\, dv + \sum_{w\in W(\mathbb{K})} \Big(\sum_{c\in\mathbb{K}^\times} \mathcal{W}_{c\varphi_0}[\eta](wsg) +$$
$$\sum_{c\in\mathbb{K}^\times}\sum_{i\in I}\sum_{\gamma\in\Gamma_i}\sum_{\psi\in\mathfrak{g}^\times_{-\alpha_i}} \mathcal{W}_{c\varphi_0+\psi}[\eta](\gamma wsg) + \sum_{\tilde\gamma\in\mathcal{M}_{V(\mathbb{A})}} \int \mathcal{W}_{\varphi_0+\psi_0}[\eta](v\tilde\gamma wsg)\Big)$$

contribution from $\mathcal{O}_{ntm}$ to the abelian term

Theorem: w/ **[Gustafsson, Gourevitch, Kleinschmidt, Sahi]**

Consider $G = E_8$. *Any automorphic form* $\eta \in \pi_{ntm}$ *may then be expanded as follows*

$$\eta(g) = \mathcal{F}_{S_\alpha,0}[\eta](g) + \sum_{\tilde\gamma \in \Gamma_{\varphi_0}} \Big(\mathcal{W}_{\varphi_0}[\eta](\tilde\gamma g) + \sum_{i \in I} \sum_{\gamma \in \Gamma_i} \sum_{\psi \in \mathfrak{g}^\times_{-\alpha_i}} \mathcal{W}_{\varphi_0+\psi}[\eta](\gamma\tilde\gamma g) \Big)$$

$$+ \sum_{\tilde\gamma \in \Lambda_{V(\mathbb{A})}} \int \mathcal{W}_{\varphi_0+\psi_0}[\eta](v\tilde\gamma g)\, dv + \sum_{w \in W(\mathbb{K})} \Big(\sum_{c \in \mathbb{K}^\times} \mathcal{W}_{c\varphi_0}[\eta](wsg) +$$

$$\sum_{c \in \mathbb{K}^\times} \sum_{i \in I} \sum_{\gamma \in \Gamma_i} \sum_{\psi \in \mathfrak{g}^\times_{-\alpha_i}} \mathcal{W}_{c\varphi_0+\psi}[\eta](\gamma wsg) + \sum_{\tilde\gamma \in \mathcal{M}_{V(\mathbb{A})}} \int \mathcal{W}_{\varphi_0+\psi_0}[\eta](v\tilde\gamma wsg) \Big)$$

contribution from $\mathcal{O}_{min}$ to the non-abelian term

Theorem: w/ **[Gustafsson, Gourevitch, Kleinschmidt, Sahi]**

Consider $G = E_8$. *Any automorphic form* $\eta \in \pi_{ntm}$ *may then be expanded as follows*

$$\eta(g) = \mathcal{F}_{S_\alpha,0}[\eta](g) + \sum_{\tilde\gamma\in\Gamma_{\varphi_0}} \Big(\mathcal{W}_{\varphi_0}[\eta](\tilde\gamma g) + \sum_{i\in I}\sum_{\gamma\in\Gamma_i}\sum_{\psi\in\mathfrak{g}^\times_{-\alpha_i}} \mathcal{W}_{\varphi_0+\psi}[\eta](\gamma\tilde\gamma g)\Big)$$

$$+ \sum_{\tilde\gamma\in\Lambda} \int_{V(\mathbb{A})} \mathcal{W}_{\varphi_0+\psi_0}[\eta](v\tilde\gamma g)\, dv + \sum_{w\in W(\mathbb{K})} \Big(\sum_{c\in\mathbb{K}^\times} \mathcal{W}_{c\varphi_0}[\eta](wsg) +$$

$$\sum_{c\in\mathbb{K}^\times}\sum_{i\in I}\sum_{\gamma\in\Gamma_i}\sum_{\psi\in\mathfrak{g}^\times_{-\alpha_i}} \mathcal{W}_{c\varphi_0+\psi}[\eta](\gamma wsg) + \sum_{\tilde\gamma\in\mathcal{M}} \int_{V(\mathbb{A})} \mathcal{W}_{\varphi_0+\psi_0}[\eta](v\tilde\gamma wsg)\Big)$$

contribution from $\mathcal{O}_{ntm}$ to the non-abelian term

Theorem: w/ **[Gustafsson, Gourevitch, Kleinschmidt, Sahi]**

Consider $G = E_8$. *Any automorphic form* $\eta \in \pi_{ntm}$ *may then be expanded as follows*

$$\eta(g) = \mathcal{F}_{S_\alpha,0}[\eta](g) + \sum_{\tilde{\gamma}\in\Gamma_{\varphi_0}} \Big(\mathcal{W}_{\varphi_0}[\eta](\tilde{\gamma}g) + \sum_{i\in I}\sum_{\gamma\in\Gamma_i}\sum_{\psi\in\mathfrak{g}^\times_{-\alpha_i}} \mathcal{W}_{\varphi_0+\psi}[\eta](\gamma\tilde{\gamma}g) \Big)$$
$$+ \sum_{\tilde{\gamma}\in\Lambda_{V(\mathbb{A})}} \int \mathcal{W}_{\varphi_0+\psi_0}[\eta](v\tilde{\gamma}g)\, dv + \sum_{w\in W(\mathbb{K})} \Big(\sum_{c\in\mathbb{K}^\times} \mathcal{W}_{c\varphi_0}[\eta](wsg) +$$
$$\sum_{c\in\mathbb{K}^\times}\sum_{i\in I}\sum_{\gamma\in\Gamma_i}\sum_{\psi\in\mathfrak{g}^\times_{-\alpha_i}} \mathcal{W}_{c\varphi_0+\psi}[\eta](\gamma wsg) + \sum_{\tilde{\gamma}\in\mathcal{M}_{V(\mathbb{A})}} \int \mathcal{W}_{\varphi_0+\psi_0}[\eta](v\tilde{\gamma}wsg) \Big)$$

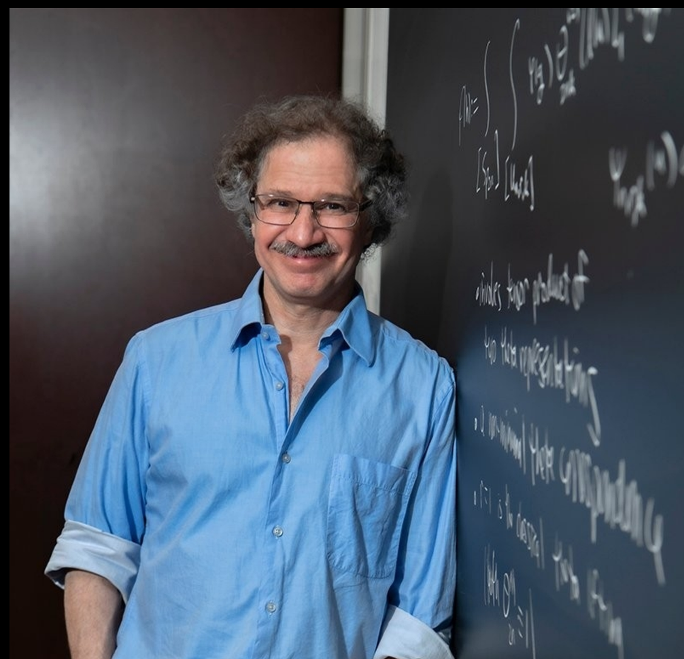
From this we can in principle extract 1/4 BPS instanton effects

[In progress with Gustafsson, Gourevitch, Kleinschmidt, Sahi]

See also [Bossard-Pioline]

4. “Generalised automorphic forms”

work in progress with



Solomon Friedberg

Axel Kleinschmidt

Dmitry Gourevitch

Guillaume Bossard

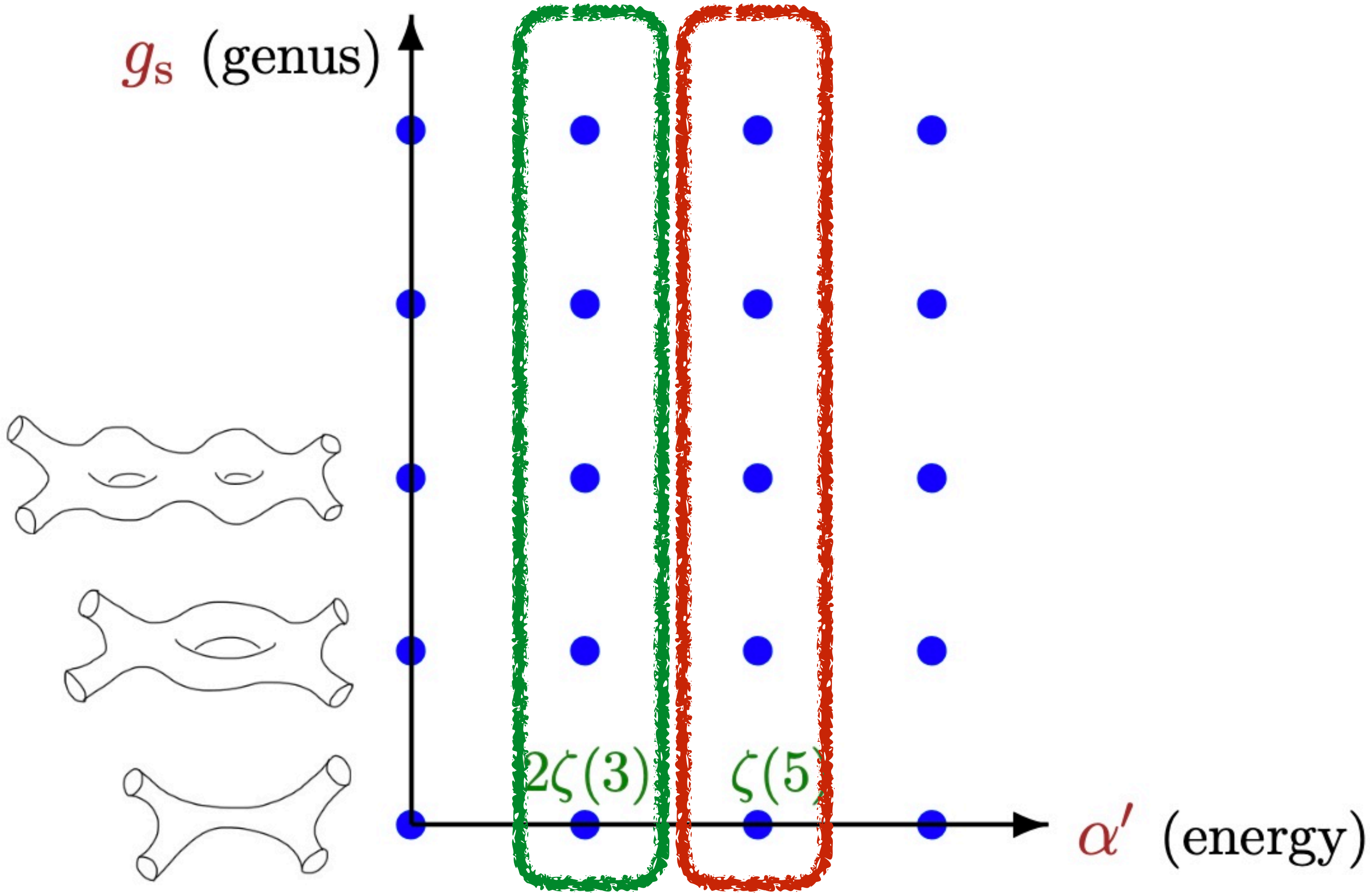
$$\int d^{10-n}x\sqrt{G}\left[(\alpha')^3 f_0(g)\mathcal{R}^4 + (\alpha')^5 f_4(g)\partial^4\mathcal{R}^4 + \cdots\right]$$

What happens at higher orders?

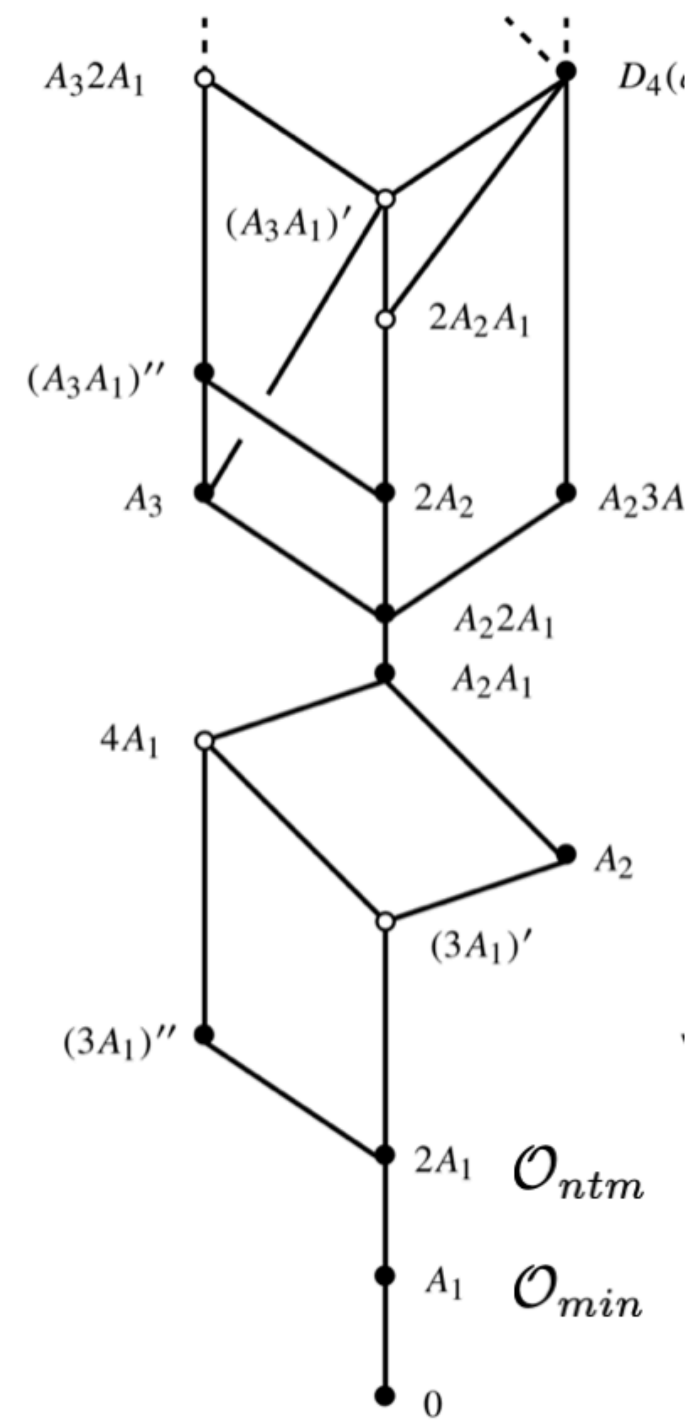
The next term is $\partial^6\mathcal{R}^4$

What kind of automorphic object appears here?

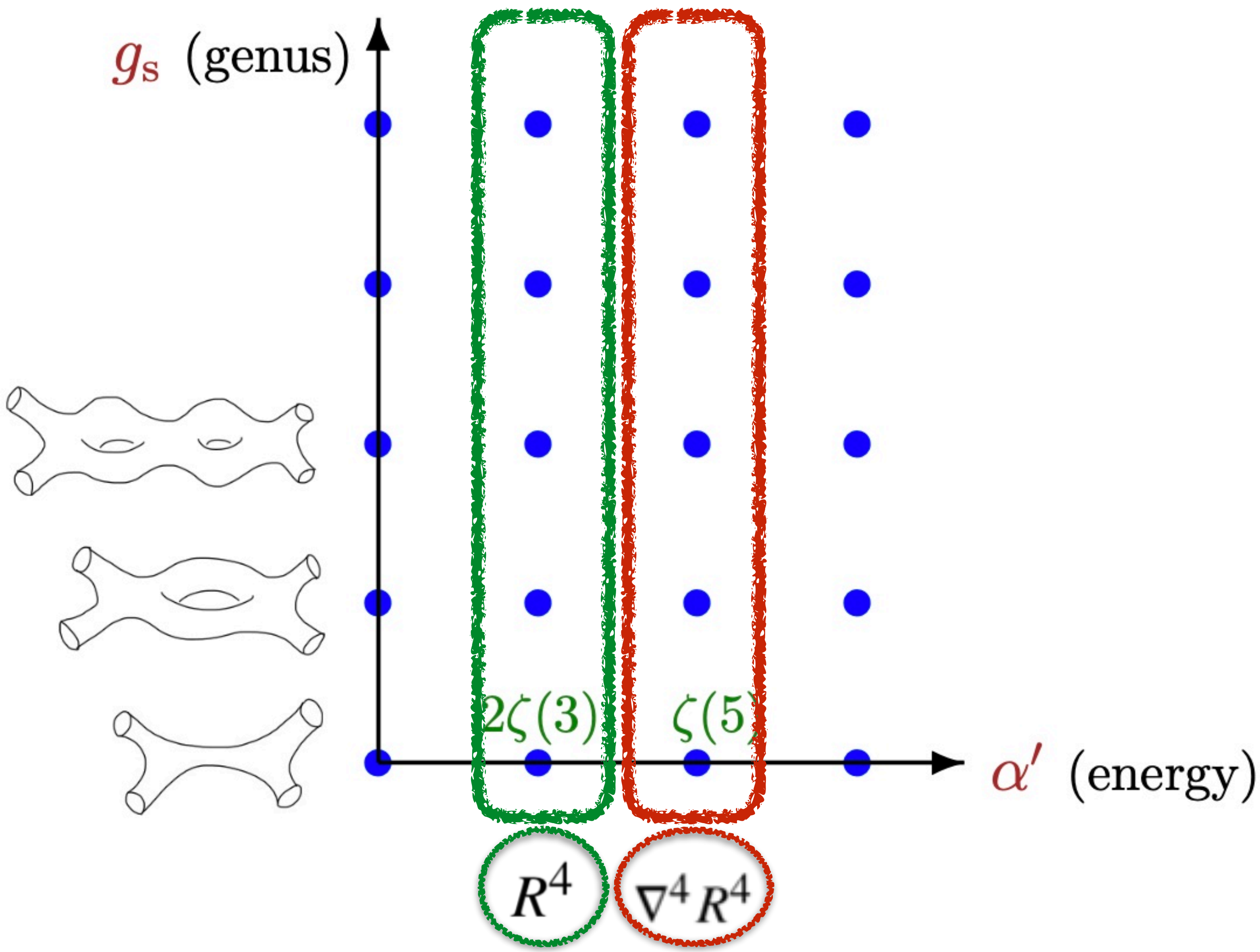
What happens at higher orders?



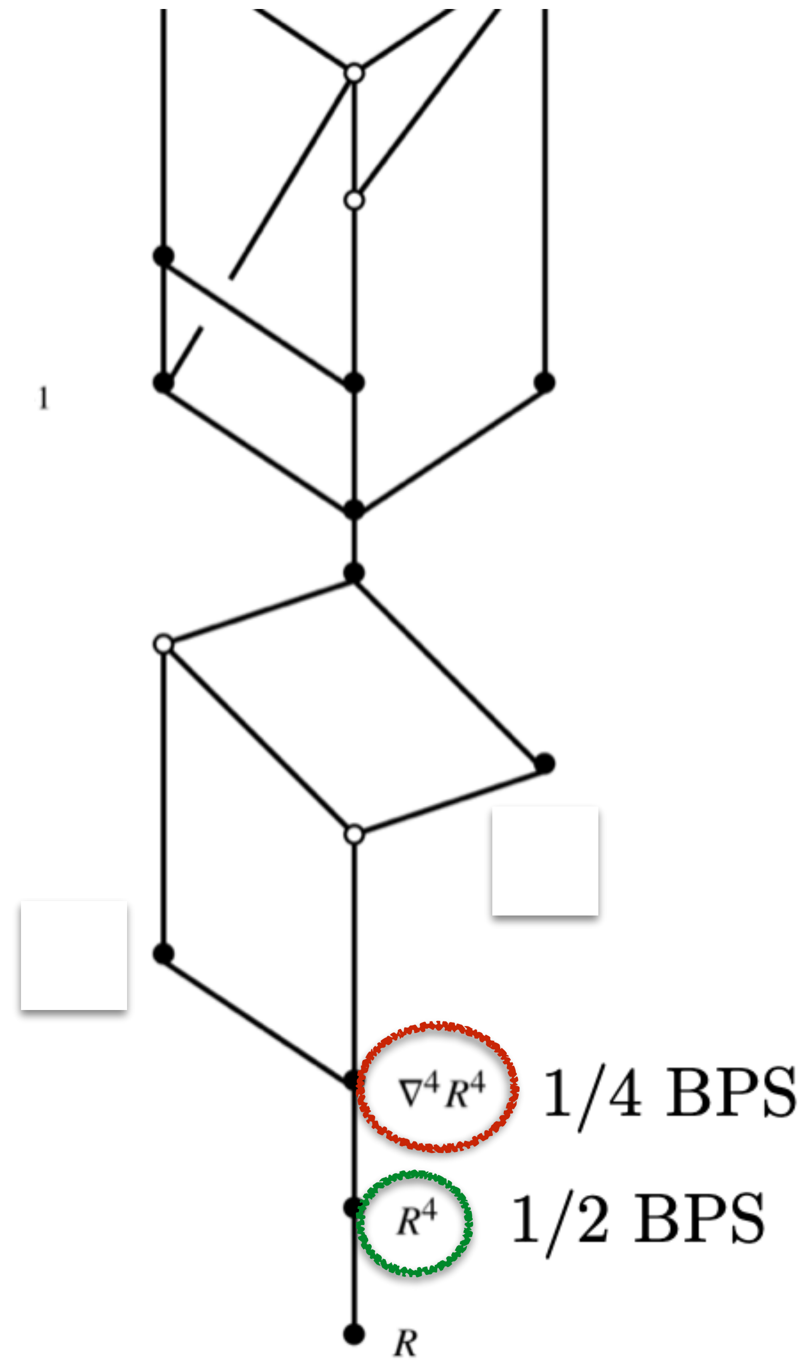
Hasse diagram for E7

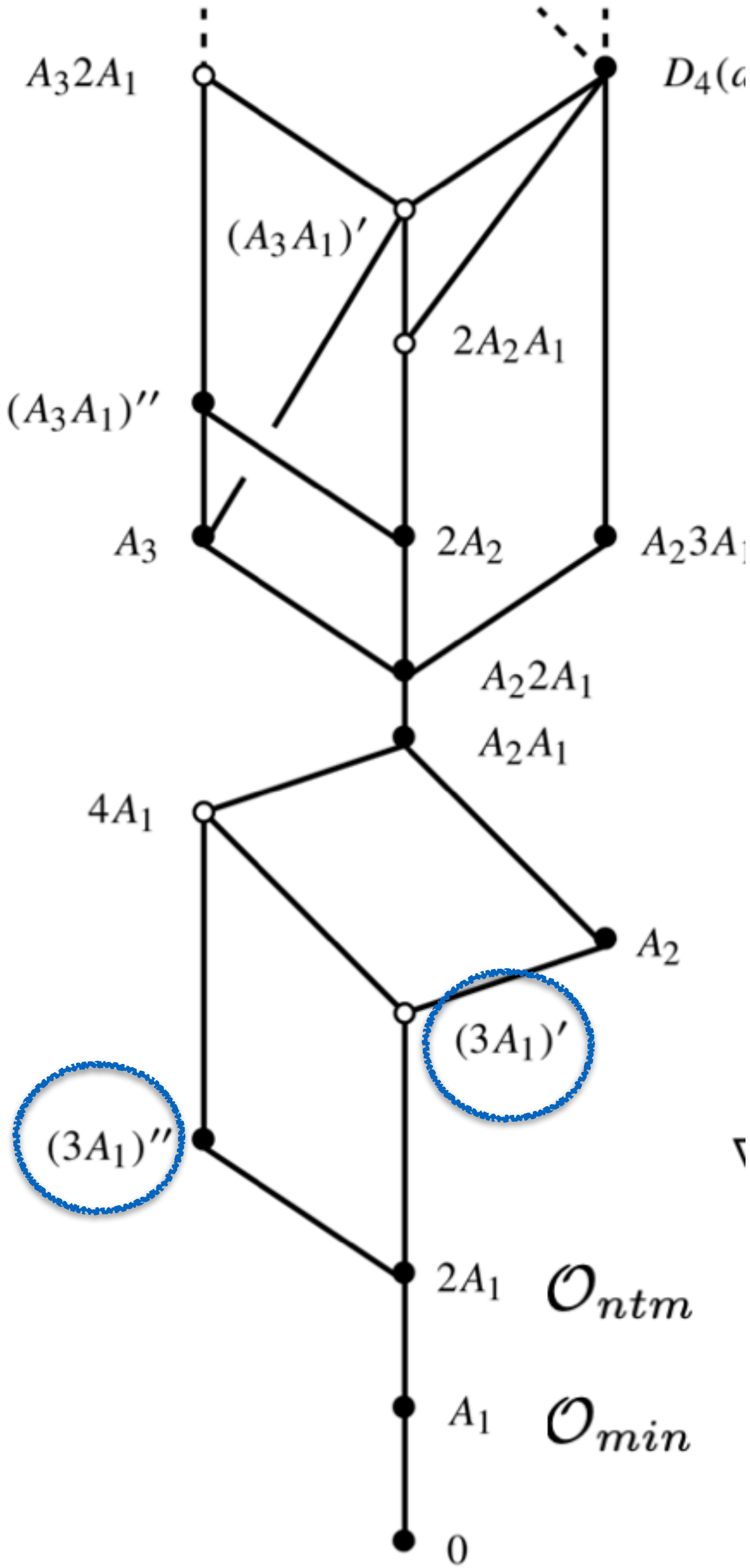


What happens at higher orders?

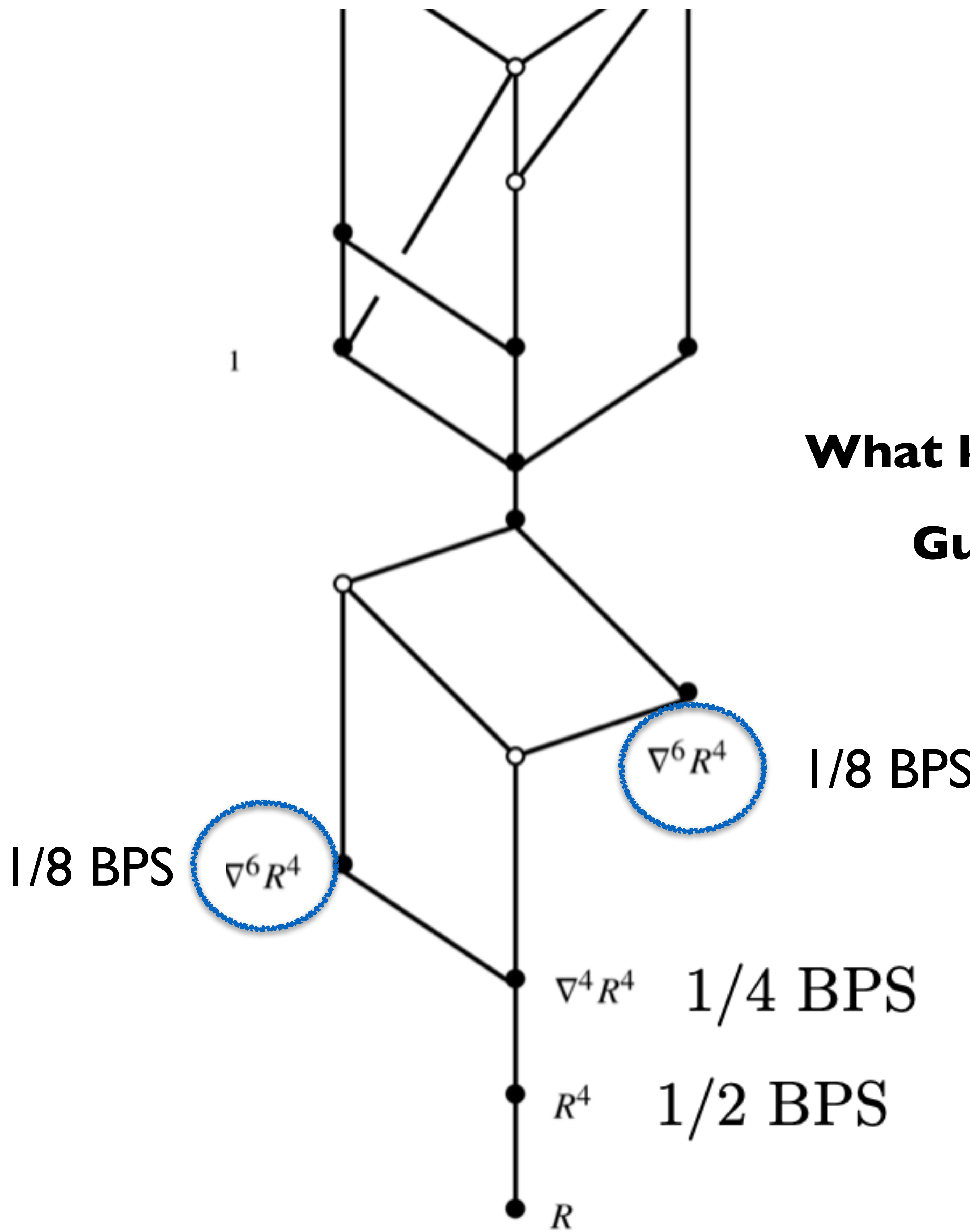


Hasse diagram for E7



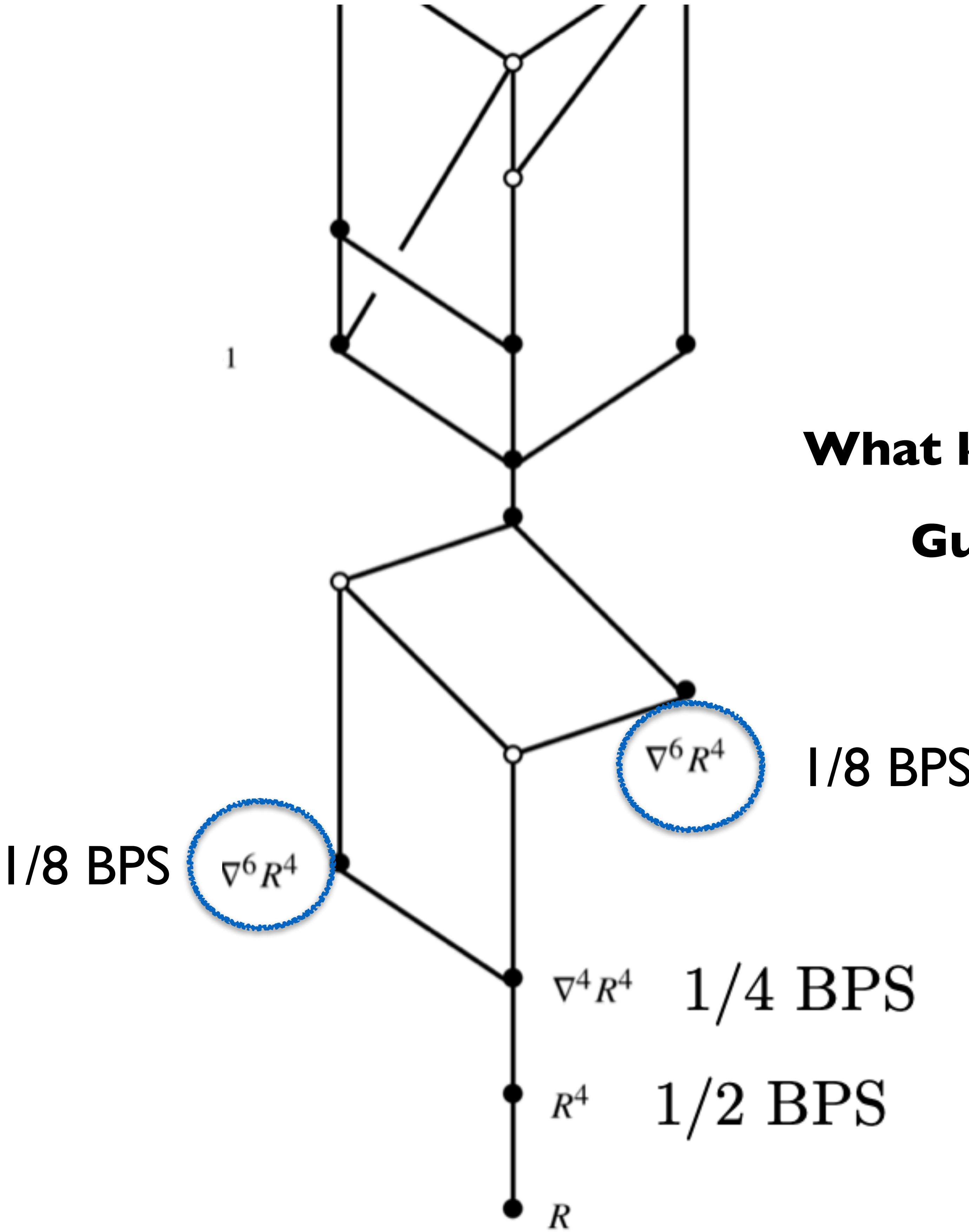


There are two maximal orbits
at the next level!



What kind of automorphic object appears?

Guess: "next-to-next-to min rep"



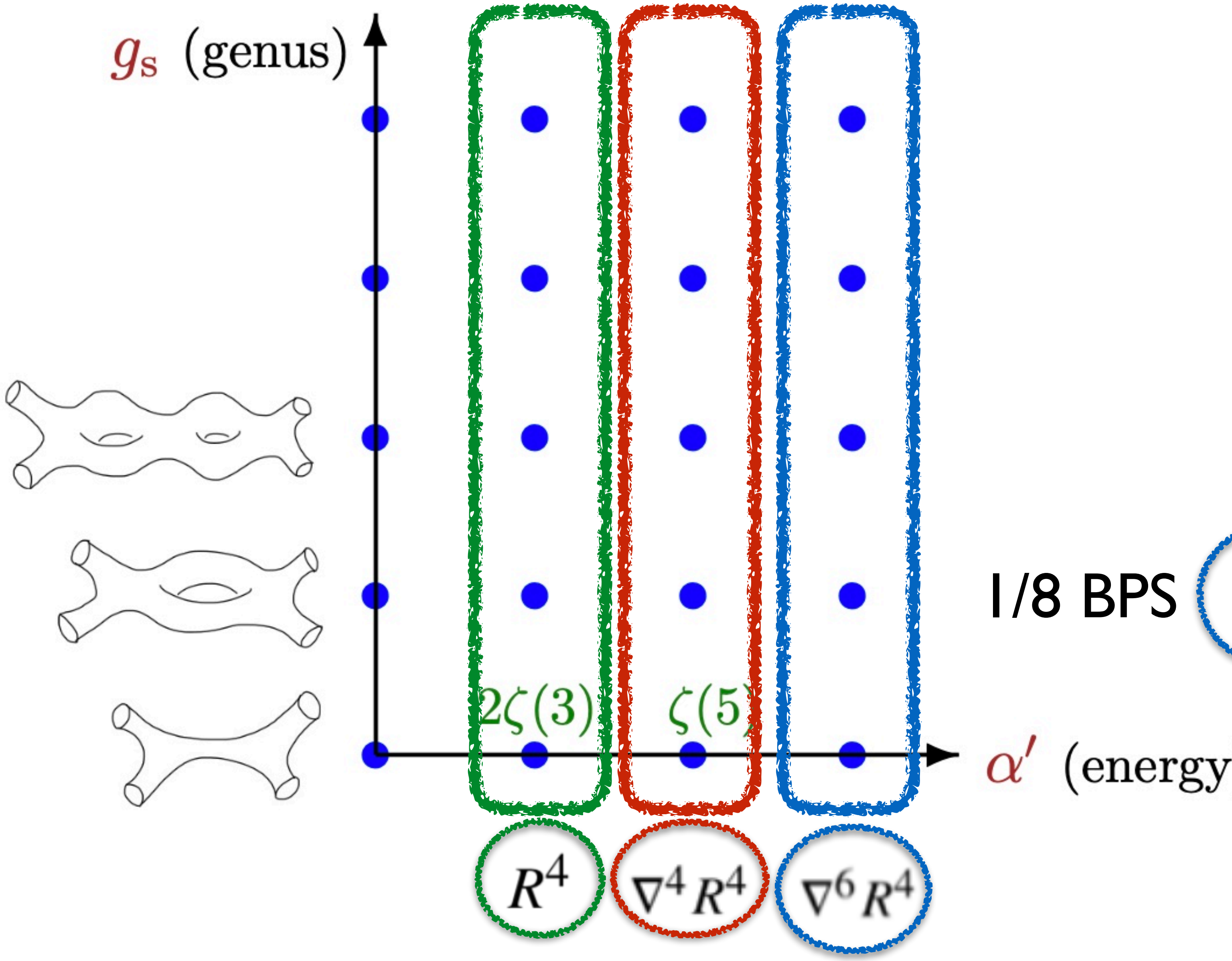
What kind of automorphic object appears?

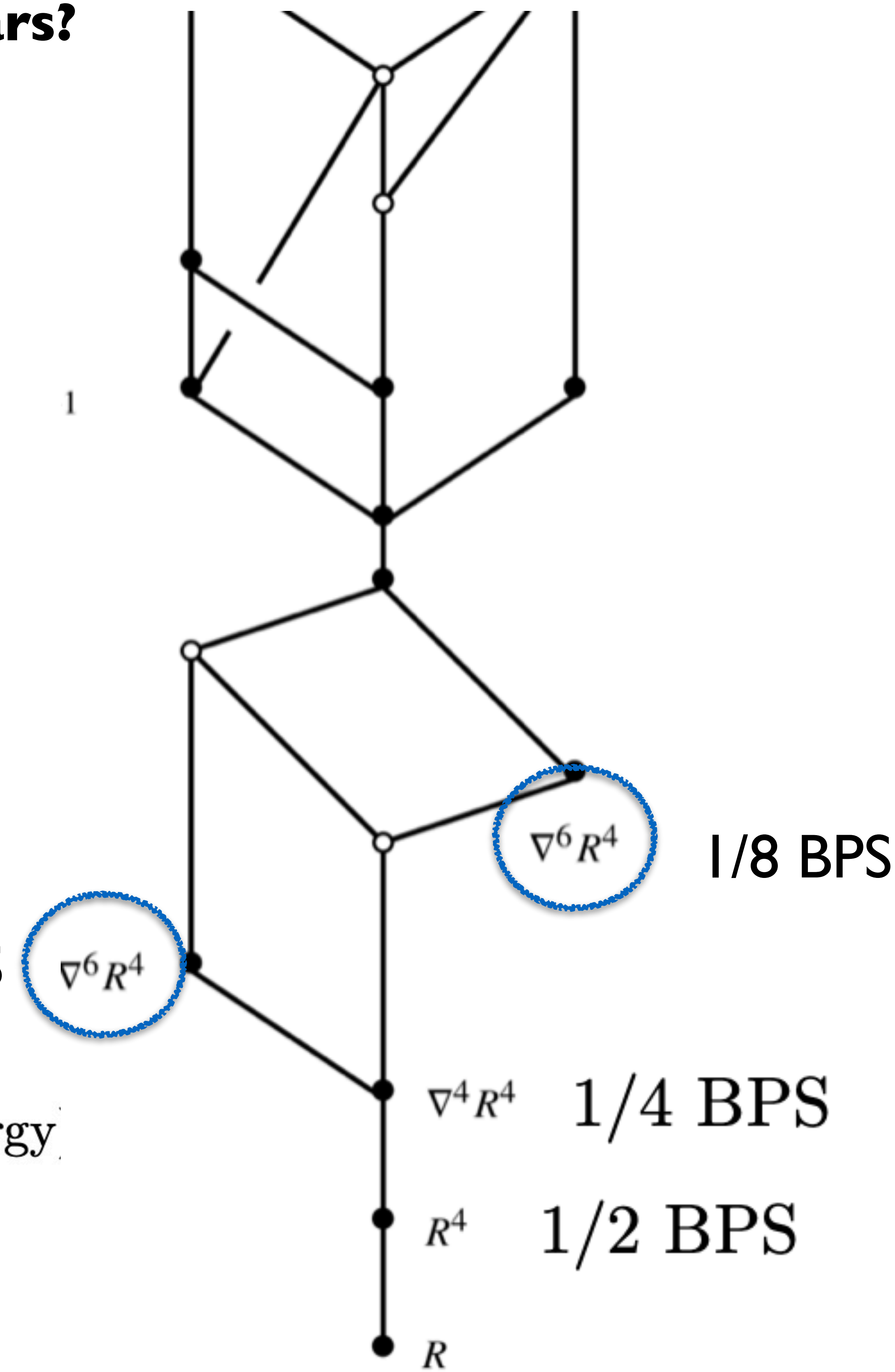
Guess: "next-to-next-to min rep"

Wrong

What kind of automorphic object appears?

Guess: **“next-to-next-to min rep”** Wrong





For example, $\partial^6\mathcal{R}^4$ in D=10 predicts a function $\mathcal{F}(\tau)$ satisfying

$$\big(\Delta_\tau - 12\big)\mathcal{F}(\tau) = -\big(E_{3/2}(\tau)\big)^2$$

[Green, Vanhove][Green, Miller, Vanhove][Fedosova, Klinger-Logan, Radchenko]

For example, $\partial^6\mathcal{R}^4$ in D=10 predicts a function $\mathcal{F}(\tau)$ satisfying

$$(\Delta_\tau - 12)\mathcal{F}(\tau) = -\big(E_{3/2}(\tau)\big)^2$$

[Green, Vanhove][Green, Miller, Vanhove][Fedosova, Klinger-Logan, Radchenko]

These are not eigenfunctions of the Laplace operators, but satisfy Poisson-type equations.

(Violates $\mathcal{Z}$ -finiteness)

The equation can be solved using Poincaré series, Fourier expansion or spectral methods

[Green, Miller, Vanhove][Ahlén, Kleinschmidt][Dorigoni, Kleinschmidt][Klinger-Logan] [Dorigoni, Green, Wen]…

More generally the equation is of the form

$$(\Delta_{G/K} - \lambda) f_{n^2 tm} = -f_{min}^2$$

More generally the equation is of the form

$$(\Delta_{G/K} - \lambda) f_{n^2tm} = -f_{min}^2$$

Explicit solution proposed for D=6

$$f_{n^2tm} = 8\pi \,\mathrm{R.N.} \int_{\mathcal{F}_2} \frac{\mathrm{d}^6\Omega}{|\Omega_2|^3} \Gamma_{5,5,2}(\Omega,\phi)\, \varphi_{\mathrm{KZ}}(\Omega) + \frac{16\zeta(8)}{189} \widehat{E}^{D_5}_{4\Lambda_5}$$

[Pioline][Bossard, Kleinschmidt][Bossard, Kleinschmid, Pioline]

- $SO(5,5;\mathbb{Z})$ -invariant
- Solves the differential equation
- Fourier expansion partially known

What kind of object are we dealing with?

→ f_{n^2tm}

Recall: Automorphic representation is given by

- $(\mathfrak{g}, K)$ -module at Archimedean place
- $G(\mathbb{Q}_p)$ -representation at finite places

We don't know the p-adic story yet, but we may be able to say something about the Archimedean setting

Focus on Archimedean setting. $G(\mathbb{R})$ -action given by

$$(D_X F)(g) = \frac{d}{dt} F(g e^{tX})\big|_{t=0}$$

This extends to $\mathcal{U}(\mathfrak{g})$ and gives a $(\mathfrak{g}, K)$-module in the standard case.

Focus on Archimedean setting. $G(\mathbb{R})$ -action given by

$$(D_X F)(g) = \frac{d}{dt} F(g e^{tX})\big|_{t=0}$$

This extends to $\mathcal{U}(\mathfrak{g})$ and gives a $(\mathfrak{g}, K)$-module in the standard case.

When $\mathcal{Z}$ -finiteness fails, still get a $\mathcal{U}(\mathfrak{g})$ -module!

(But **not** a $(\mathfrak{g}, K)$ -module)

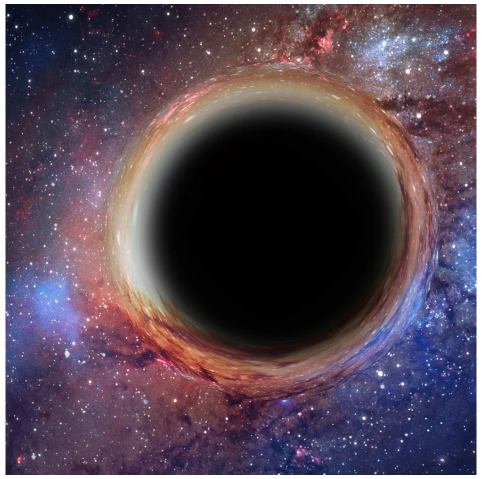
But what kind of module is this?

Conjecture: The Fourier coefficients are of the form

$$\int_{U(\mathbb{Z})\backslash U(\mathbb{R})} f_{n^2tm}(ug)\overline{\psi(u)}du = c_\psi(f_{n^2tm})B_{hom,\psi}(g) + B_\psi^{min^2}(g)$$

U Heisenberg → $B_\psi^{min^2}(g)$ exponentially suppressed

Physics predicts that $c_\psi(f_{n^2tm})$ blows up exponentially due to **black holes**



All this is work in progress with Bossard, Friedberg, Gourevitch and Kleinschmidt.

To be continued…

Thank you!