Z-infinite automorphic forms?

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TAGG research day

May 8, 2026

Based on work in progress with

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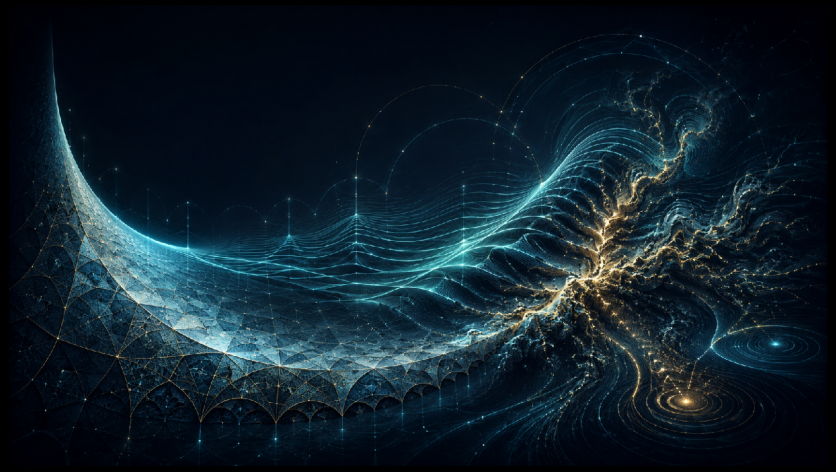
Outline



1. **Automorphic forms and representations**

2. **Motivation from amplitudes**

3. **Z-infinite automorphic forms?**

2. Automorphic forms and representations

Data:

- $G(\mathbb{R})$ real simple Lie group (e.g. $SL(n,\mathbb{R})$)
- $G(\mathbb{Z}) \subset G$ arithmetic subgroup (e.g. $SL(n,\mathbb{Z})$)

Definition:

An **automorphic form** is a smooth function $\varphi : G \longrightarrow \mathbb{C}$ satisfying

1. Automorphy: $\forall \gamma \in G(\mathbb{Z}), \;\; \varphi(\gamma g) = \varphi(g)$
2. φ is an eigenfunction of the ring of inv. diff. operators on G ($\mathcal{Z}$-finiteness)
3. φ has well-behaved growth conditions
4. K-finiteness

Non-holomorphic Eisenstein series

Consider the sum:

$$E_s(\tau) = \sum_{(c,d)=1} \frac{y^s}{|c\tau + d|^{2s}}$$

non-holomorphic Eisenstein series

$s \in \mathbb{C}$

→ a function on $\mathbb{H} = \{\tau = x + iy \in \mathbb{C} \,|\, y > 0\} \cong SL(2,\mathbb{R})/SO(2)$

→ invariant under $\tau \mapsto \gamma \cdot \tau = \dfrac{a\tau + b}{c\tau + d}$

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$$

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→ invariant under $\tau \mapsto \gamma \cdot \tau = \frac{a\tau + b}{c\tau + d}$

→ converges absolutely for $\Re s > 1$

→ $\Delta_{\mathbb{H}} E_s = s(s-1)E_s$

Can be lifted to a function on $SL(2,\mathbb{R})$

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Lie algebra

$\mathfrak{sl}(2,\mathbb{R})$ $\quad [h,e] = 2e, \quad [h,f] = -2f, \quad [e,f] = h$

Basis of $\mathcal{U}(\mathfrak{sl}(2,\mathbb{R}))$ $\quad \{f^i h^j e^k \,|\, i,j,k \geq 0\}$ $\quad$ universal enveloping algebra

Centre $\mathcal{Z}$

Generated by the **Casimir**

$$\Omega = h^2 + 2ef + 2fe$$

$\mathcal{Z} = \mathbb{R}[\Omega]$

polynomial ring

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On

$$SL(2,\mathbb{R})/SO(2) \cong \mathbb{H}$$

$$\Omega \leftrightarrow \Delta_{\mathbb{H}}$$

Eisenstein series on semi-simple Lie groups

(Borel) **Eisenstein series** on a semi-simple Lie group is defined by:

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Iwasawa decomposition: $G = BK = NAK$

$$A \sim \begin{pmatrix} * & & \\ & * & \\ & & * \end{pmatrix} \qquad N \sim \begin{pmatrix} 1 & * & * \\ & 1 & * \\ & & 1 \end{pmatrix}$$

Adelic picture

$$G(\mathbb{A}) = G(\mathbb{R}) \times \prod_{p<\infty}' G(\mathbb{Q}_p)$$

Definition (Automorphic form) An *automorphic form* is a smooth function $\varphi\colon G(\mathbb{Q})\backslash G(\mathbb{A}) \to \mathbb{C}$ satisfying the following conditions:

(1) *left $G(\mathbb{Q})$-invariance:* $\varphi(\gamma g) = \varphi(g), \qquad \gamma \in G(\mathbb{Q})$

(2) *right K-finiteness:* $\dim_{\mathbb{C}}\langle \varphi(gk) | k \in K_{\mathbb{A}} \rangle < \infty$

(3) *$\mathcal{Z}(\mathfrak{g}_{\mathbb{C}})$-finiteness:* $\dim_{\mathbb{C}}\langle X\varphi(g) | X \in \mathcal{Z}(\mathfrak{g}_{\mathbb{C}}) \rangle < \infty$

(4) *φ is of moderate growth:* for any norm $||\cdot||$ on $G(\mathbb{A})$ there exists a positive integer n and a constant C such that $|\varphi(g)| \leq C||g||^n$.

Space of automorphic forms: $\mathcal{A}(G(\mathbb{Q})\backslash G(\mathbb{A}))$

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Instead we have the following structure:

- The **universal enveloping algebra** $\mathcal{U}(\mathfrak{g})$ acts by diff. operators:

$$(D_X F)(g) = \frac{d}{dt} F(g e^{tX})\big|_{t=0}$$

- The right translations of the **finite adeles**

$$\pi(g_f) \cdot F(g') = F(g' g_f) \qquad g' \in G(\mathbb{A}), \quad g_f \in G(\mathbb{A}_f)$$

- The right translations by K_∞

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don't commute

These two actions satisfy a **compatibility condition**

$$D_X \cdot \pi(k_\infty) = \pi(k_\infty) \cdot D_{k_\infty^{-1} X k_\infty}$$

$(\mathfrak{g}, K)-$ module

(Harish-Chandra module)

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(Harish-Chandra module)

With this preparation we arrive at:

Definition (automorphic representation) A representation π of $G(\mathbb{A})$ is called an *automorphic representation* if it occurs as an irreducible constituent in the decomposition of $\mathcal{A}(G(\mathbb{Q})\backslash G(\mathbb{A}))$ with respect to the simultaneous action by

$$(\mathfrak{g}_\infty, K_\infty) \times G(\mathbb{A}_f),$$

where K_∞ and $G(\mathbb{A}_f)$ act by right-translation and $\mathfrak{g}_\infty$ by differential operators at the archimedean place.

1. Motivation from amplitudes

QFT amplitudes

Compute the probability of a scattering process between elementary particles

$$\mathcal{A}(p_1, p_2, p_3, p_4) \in \mathbb{C}$$

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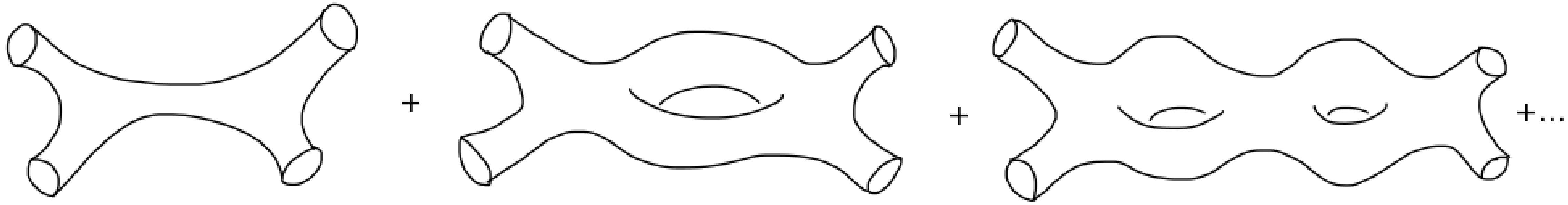
Perturbation theory: expand for small coupling g

$$\mathcal{A}(p_1, p_2, p_3, p_4) = \sum_{n \geq 0} g^n \mathcal{A}_n(p_1, p_2, p_3, p_4) + \mathcal{A}^{\text{non-pert.}}(p_1, p_2, p_3.p_4)$$

$$\sim e^{-1/g}$$

String amplitudes

Understand the structure of **string interactions**



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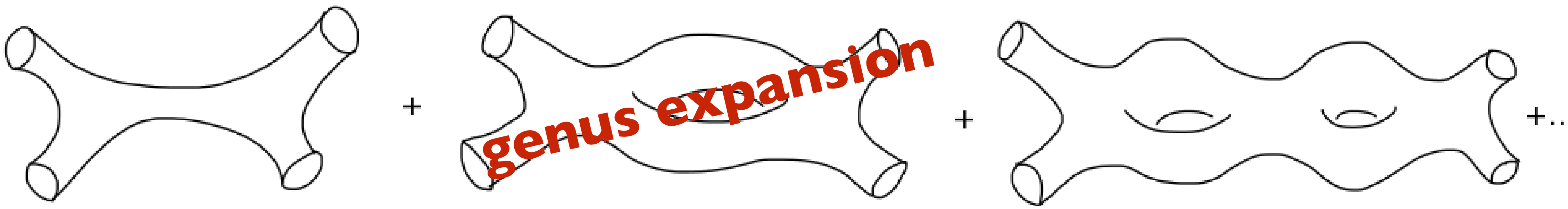
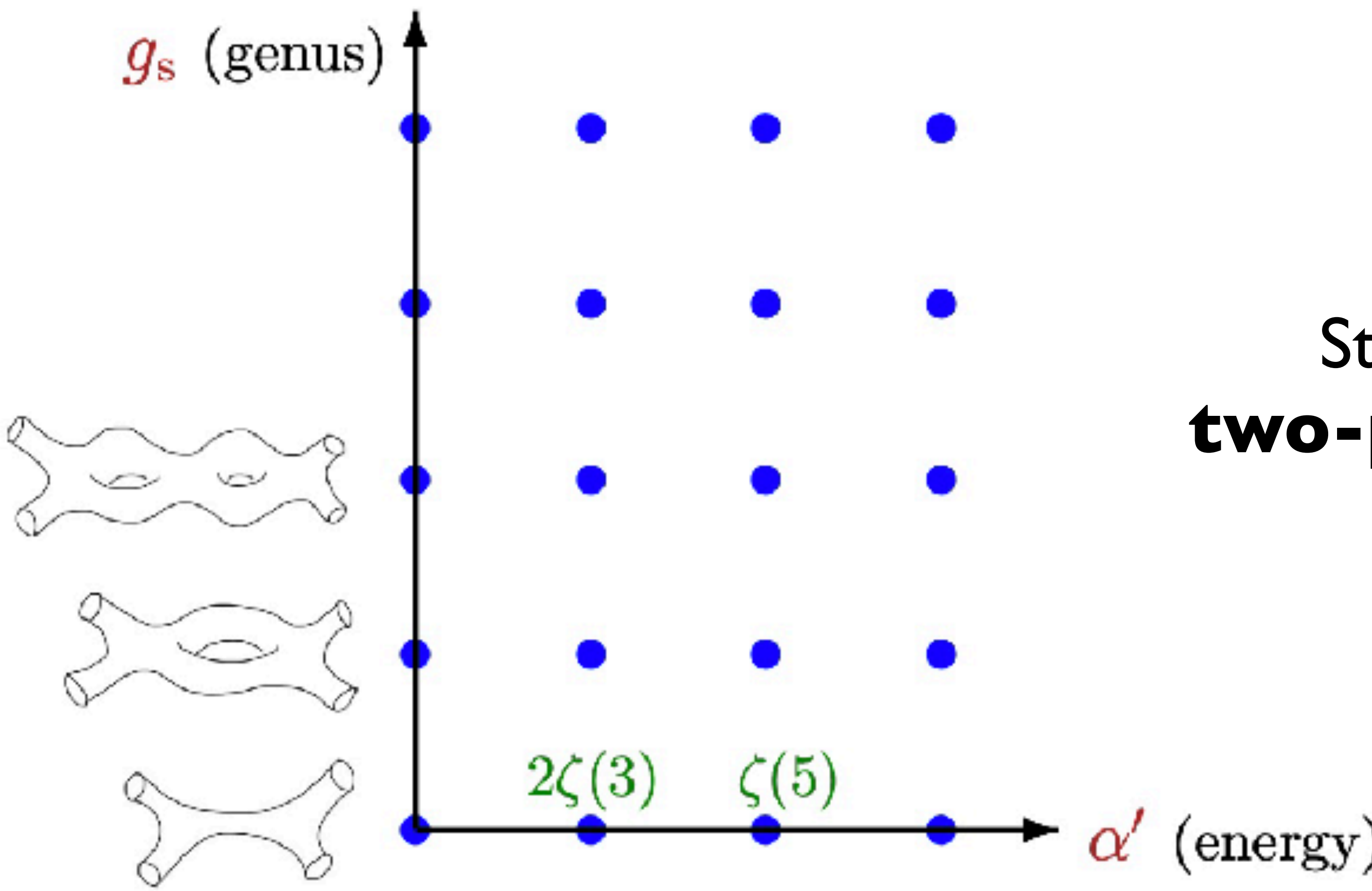
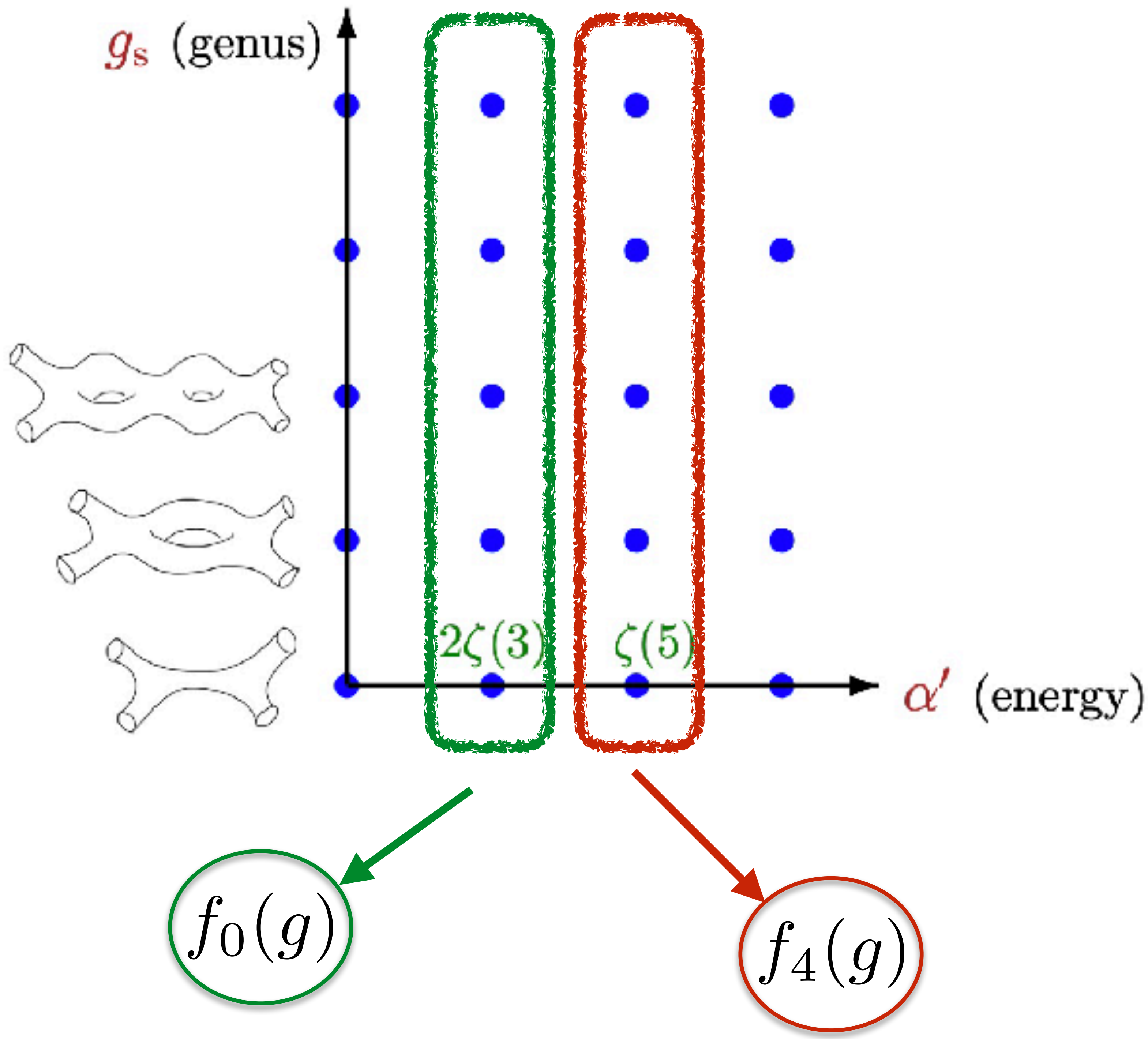
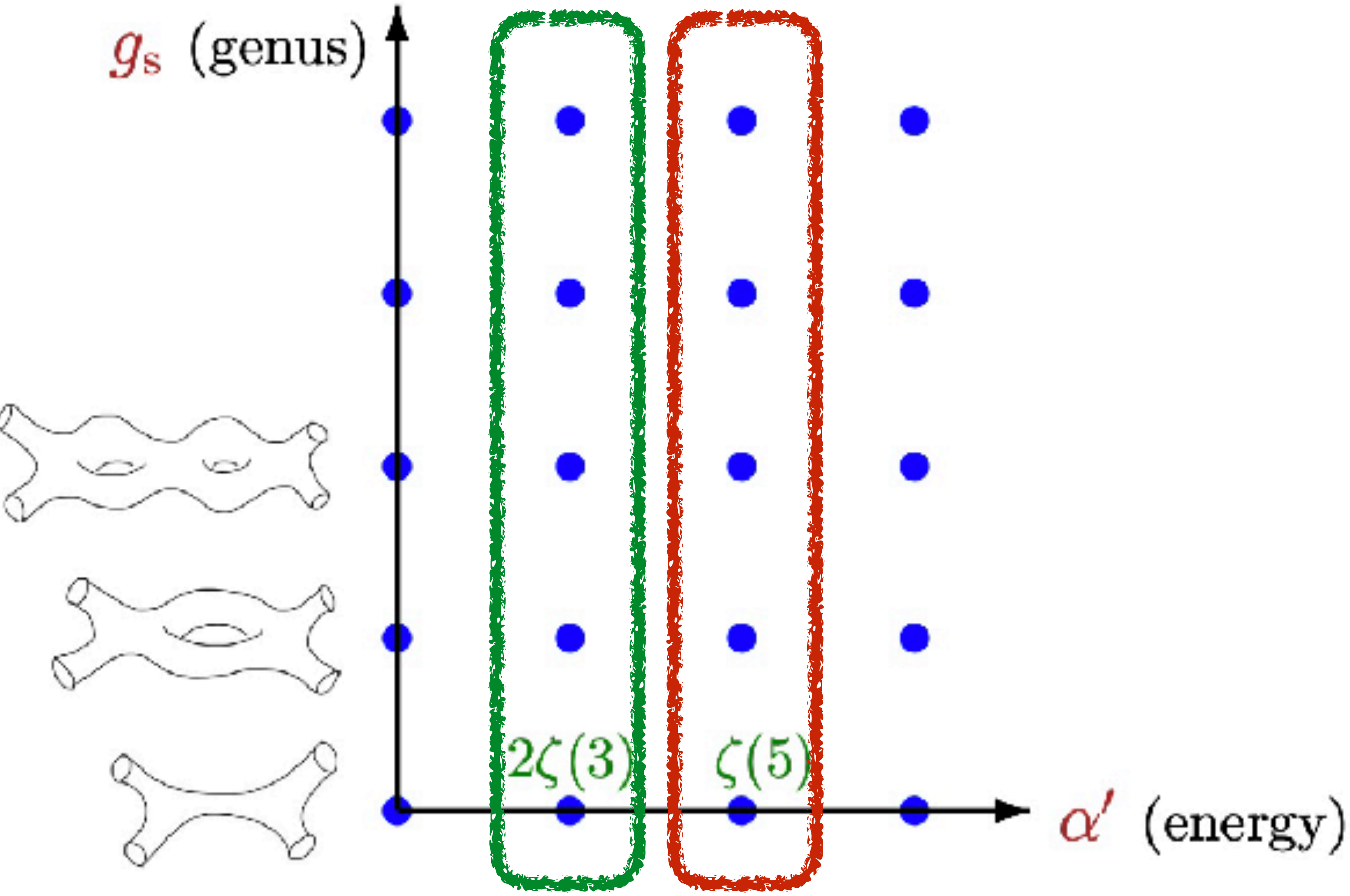




String amplitudes have a
two-parameter expansion



g_s (genus)
α' (energy)
$2\zeta(3)$
$\zeta(5)$
$f_0(g)$
$f_4(g)$

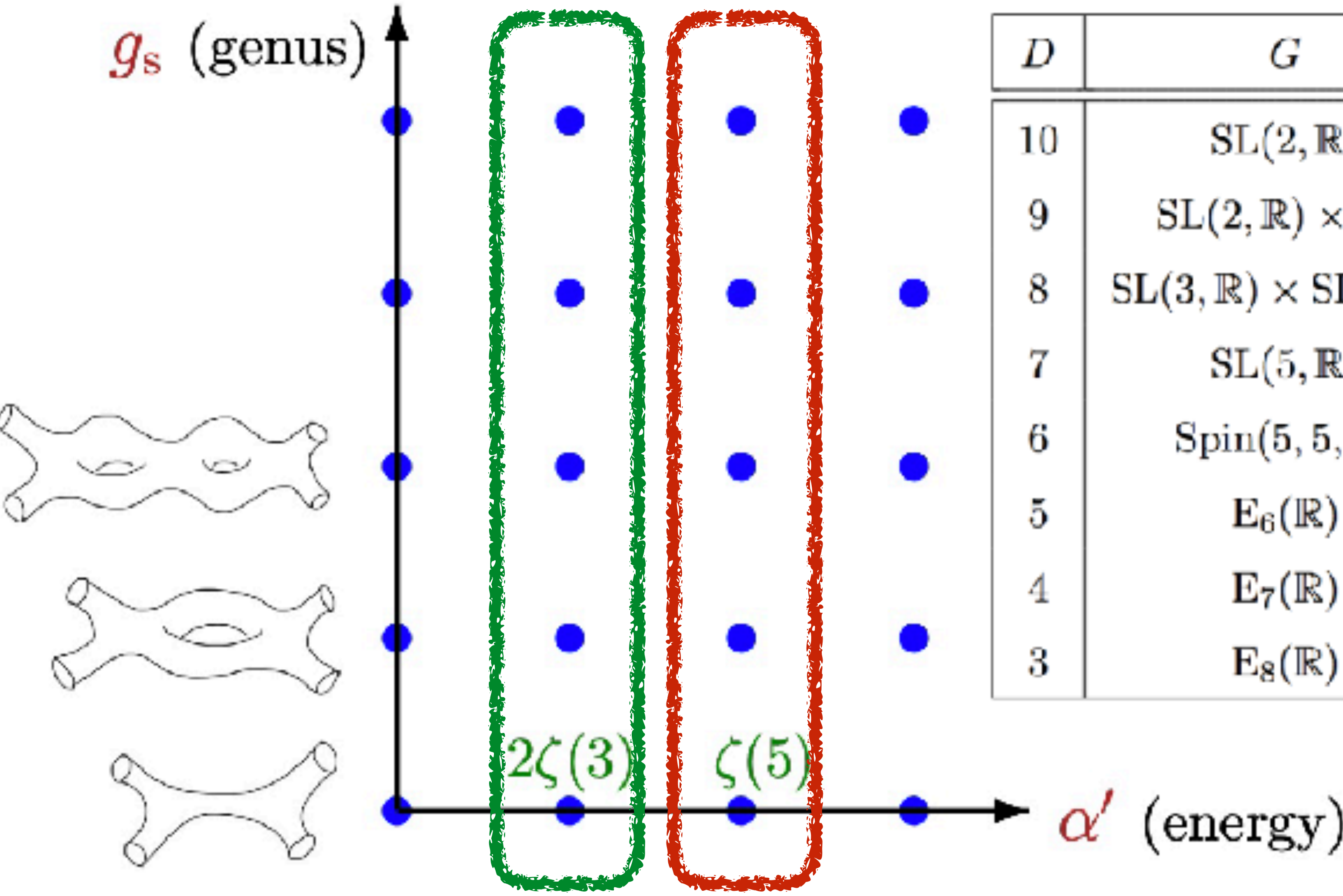


→ must be **invariant** under U-duality $G(\mathbb{Z})$

→ must be **invariant** under K

→ supersymmetry requires differential equations

→ well-defined **weak-coupling expansions** as $g_s \to 0$



D	G	K	$G(\mathbb{Z})$
10	$SL(2,\mathbb{R})$	$SO(2)$	$SL(2,\mathbb{Z})$
9	$SL(2,\mathbb{R}) \times \mathbb{R}^+$	$SO(2)$	$SL(2,\mathbb{Z})$
8	$SL(3,\mathbb{R}) \times SL(2,\mathbb{R})$	$SO(3) \times SO(2)$	$SL(3,\mathbb{Z}) \times SL(2,\mathbb{Z})$
7	$SL(5,\mathbb{R})$	$SO(5)$	$SL(5,\mathbb{Z})$
6	$Spin(5,5,\mathbb{R})$	$(Spin(5) \times Spin(5))/\mathbb{Z}_2$	$Spin(5,5,\mathbb{Z})$
5	$E_6(\mathbb{R})$	$USp(8)/\mathbb{Z}_2$	$E_6(\mathbb{Z})$
4	$E_7(\mathbb{R})$	$SU(8)/\mathbb{Z}_2$	$E_7(\mathbb{Z})$
3	$E_8(\mathbb{R})$	$Spin(16)/\mathbb{Z}_2$	$E_8(\mathbb{Z})$

defining properties of an **automorphic form!**

- → must be **invariant** under U-duality $G(\mathbb{Z})$
- → must be **invariant** under K
- → supersymmetry requires differential equations
- → well-defined **weak-coupling expansions** as $g_s \to 0$

Amplitudes correspond to special representations

Borel Eisenstein series: $E(\lambda, g)$

Consider the **special locus**: $E(s,g) = E(2s\Lambda_1 - \rho, g)$

We then have:

$$\boxed{f_0(g) = E(3/2, g)}$$

$$\boxed{f_4(g) = E(5/2, g)}$$

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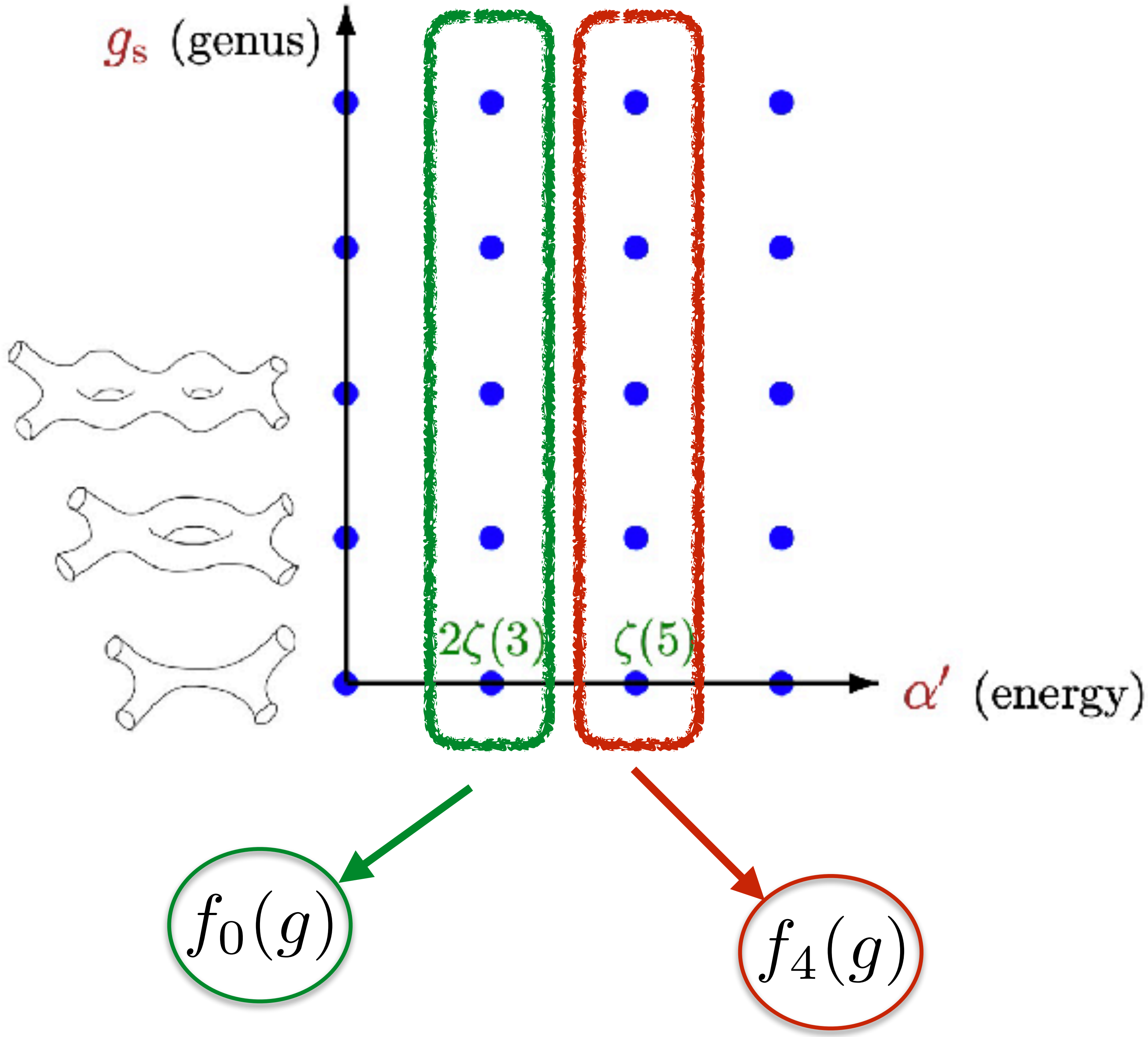
These functions are Eisenstein series attached to **small automorphic representations** of G.

[Ginzburg, Rallis, Soudry][Green, Miller, Vanhove][Pioline]

minimal automorphic representation π_{min}	**next-to-minimal automorphic representation** π_{ntm}
1/2 - BPS	1/4 - BPS

4. Z-infinite automorphic forms



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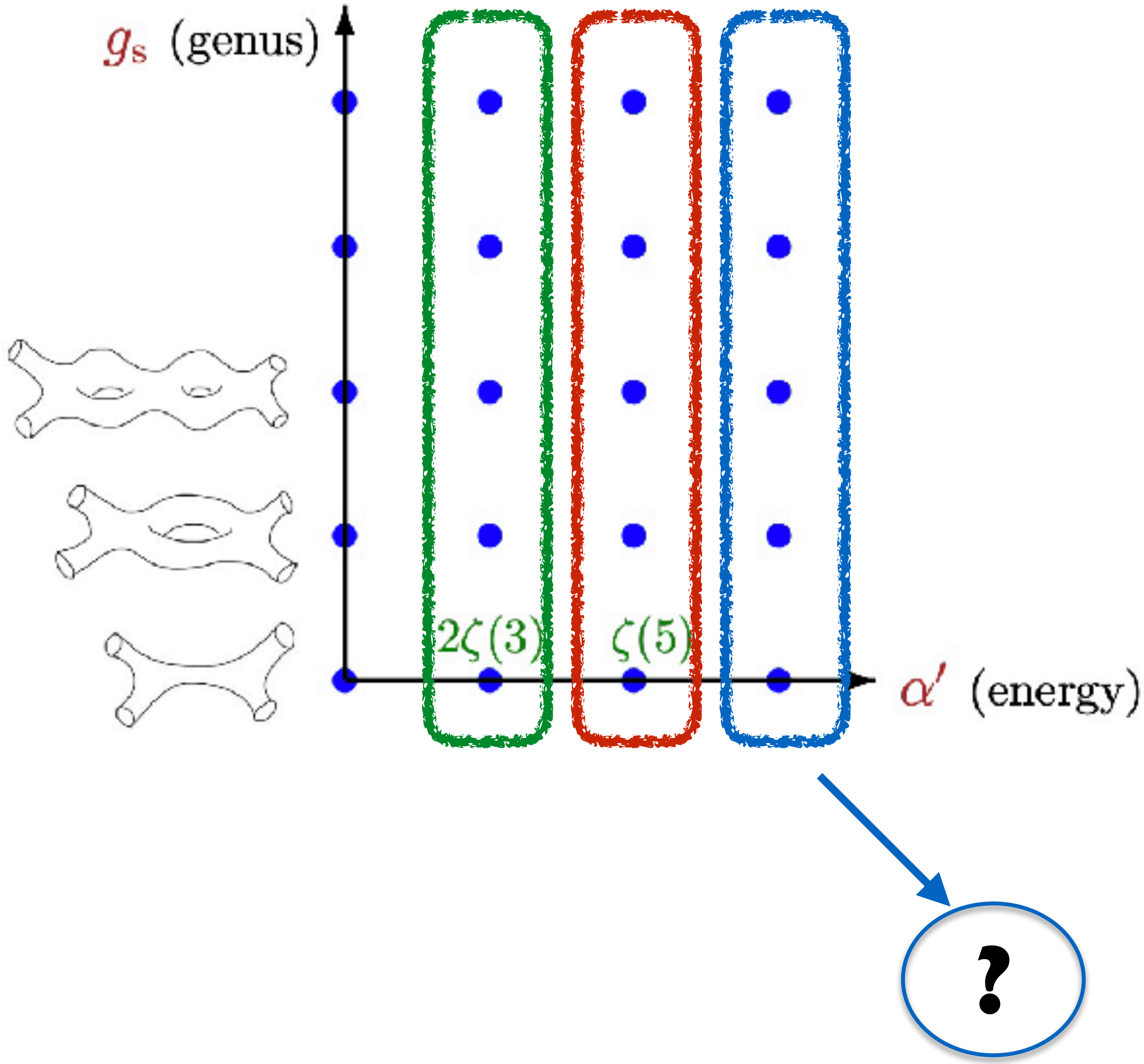
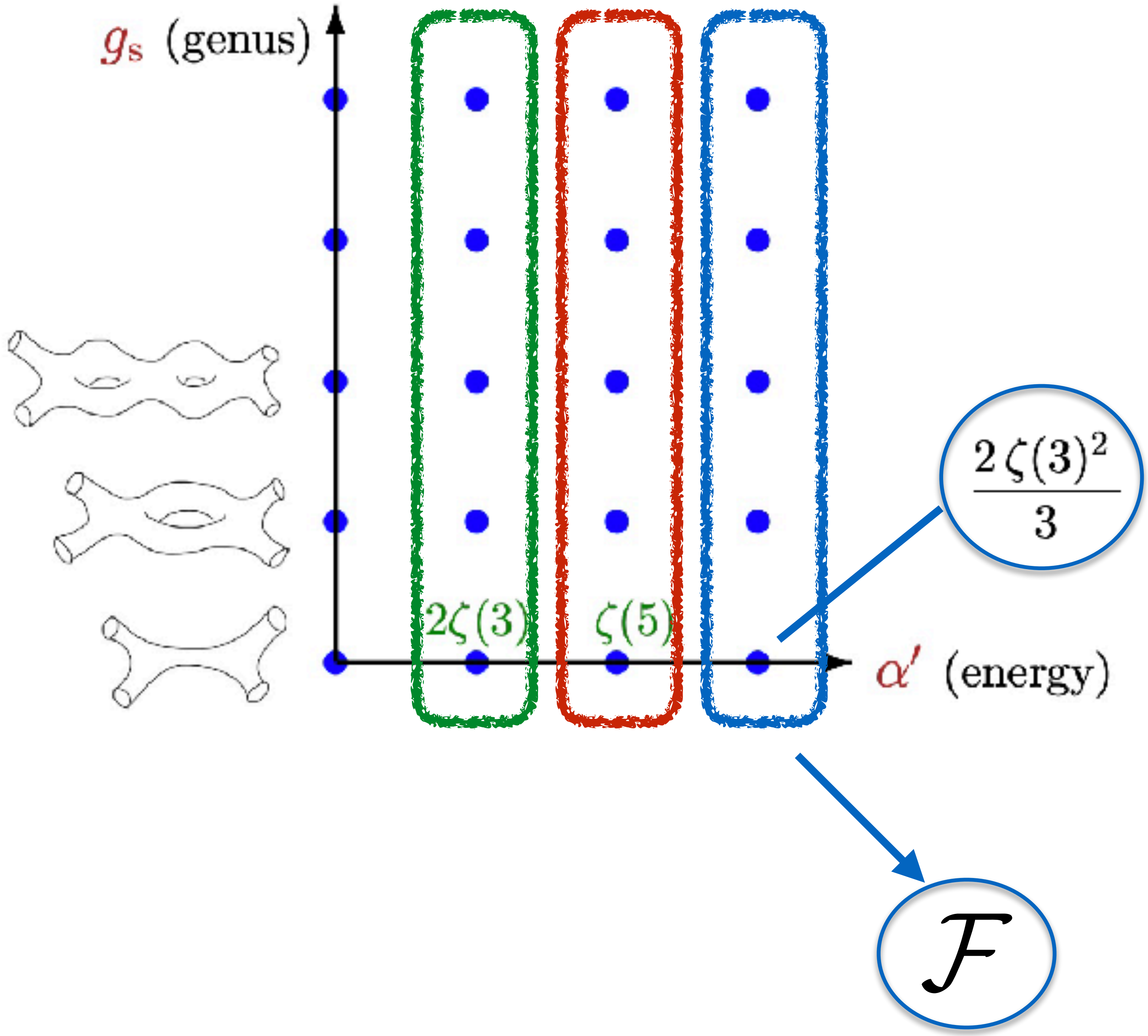


What happens at higher orders?

What kind of automorphic object appears here?



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For $SL(2,\mathbb{R})$ the function $\mathcal{F}(\tau)$ satisfies the equation

$$\left(\Delta_\tau - 12\right)\mathcal{F}(\tau) = -\left(E_{3/2}(\tau)\right)^2$$

[Green, Vanhove][Green, Miller, Vanhove][Fedosova, Klinger-Logan, Radchenko]

Where the RHS is the square of the s=3/2 Eisenstein series:

$$E_s(\tau) = \sum_{(c,d)=1} \frac{y^s}{|c\tau + d|^{2s}}$$

$$\mathcal{F}(\tau) = \sum_{\gamma \in B(\mathbb{Z})\backslash SL(2,\mathbb{Z})} \Phi(\tau)\Big|_\gamma$$

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The unique $SL(2,\mathbb{Z})$ - invariant solution is:

$$\mathcal{F}(\tau) = \sum_{\gamma \in B(\mathbb{Z})\backslash SL(2,\mathbb{Z})} \Phi(\tau)\Big|_\gamma$$

$$\Phi(\tau) = \frac{2}{3}\zeta(3)^2 y^3 + \frac{1}{9}\pi^2\zeta(3) y + \sum_{n\neq 0} c_n(y) e^{2\pi i n x}$$

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Constant term:

$$\int_{N(\mathbb{Z})\backslash N(\mathbb{R})} \mathcal{F}(ng)dn = \frac{2\zeta(3)^2}{3}y^3 + \frac{4\zeta(2)\zeta(3)}{3}y + \frac{4\zeta(4)}{y} + \frac{4\zeta(6)}{27y^3}$$

$$+y^{-2}\sum_{n=1}^{\infty} a_n e^{-4\pi n y} + \cdots$$

More generally the equation is of the form

$$(\Delta_{G/K} - \lambda)\mathcal{F} = \mathcal{S}^2$$

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- Poisson type equation with a source term
- The source $\mathcal{S}$ is an automorphic form in the **minimal representation** of G
- Violates $\mathcal{Z}$ -finiteness
- Still get a $\mathcal{U}(\mathfrak{g})$ -module! (But **not** a $(\mathfrak{g}, K)$ -module)

But what kind of module is this?

Not an automorphic form in the strict sense

In progress w/ Bossard, Friedberg, Gourevitch, Kleinschmidt

Theorem: The module generated by $\mathcal{F}$ fits into the SES sequence

$$0 \to \mathcal{U}(\mathfrak{g})\mathcal{S} \to \mathcal{U}(\mathfrak{g})\mathcal{F} \to \mathcal{U}(\mathfrak{g})E_P(s,g) \to 0$$

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- $E_P(s)$ is the homogeneous solution (an Eisenstein series)

$$(\Delta_{G/K} - \lambda(s))E_P(s,g) = 0$$

- s fixed so that $\lambda(s) = \lambda$

- We are working on a **multiplicity one result**, and a study of the **Fourier coefficients**

Outlook

- What about the finite places?
- Hecke algebra?
- Generalization of automorphic representation
- New L-functions?
- Non-abelian Fourier coefficients


Tack!