String scattering amplitudes and small automorphic representations

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Discrete symmetry groups arise in string theory in many contexts and several interesting physical quantities are defined on spaces that carry an action of these groups. Scattering amplitudes in particular have to be invariant under the group action and are often associated with automorphic forms. It has been noted first by Green, Miller and Vanhove as well as Pioline that these automorphic forms belong to very special automorphic representations, so-called small representations, leading to a rich interplay between mathematics and physics. This development and some open questions are sketched in this contribution.

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1 Introduction

In this introduction, the physical quantities of interest will be sketched for non-experts in very broad strokes, along with their properties. Readers primarily interested in the mathematical statements are welcome to skip to section 2. As a complementary and more detailed description, the reader may consult [1] as well as the original papers [2–7].

1.1 Interactions and scattering amplitudes

One of the basic objects of interest in any physical theory is what interactions between different objects it describes. The prototypical example is particle physics where Quantum Field Theory (QFT) is the prime conceptual framework for constructing and studying physical theories. For instance, Quantum Electrodynamics (QED) describes the interaction of electrons with photons. The former are among the basic carriers of electric charge and the latter of electromagnetic radiation such as light rays, and since electric charges are both acted upon by electromagnetic radiation and can produce electromagnetic radiation, there is a non-trivial interaction between electrons and photons. Interactions can be visualised in so-called Feynman diagrams, see Figure 1.

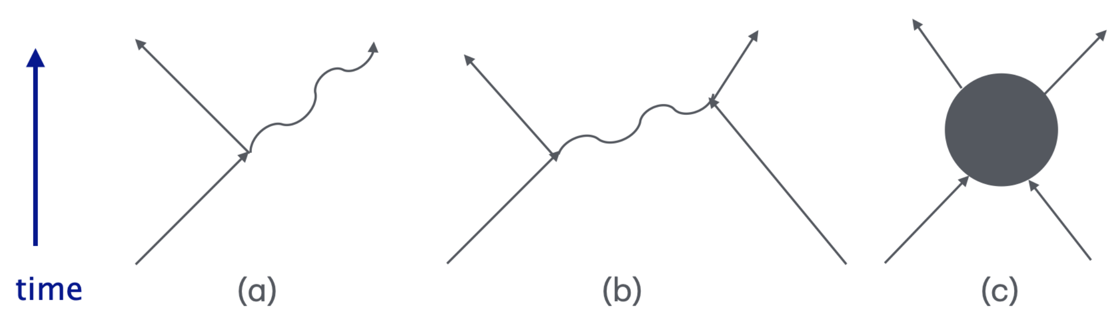

Figure 1: *Interactions in QED. (a) shows the fundamental interaction between two electrons (straight line) and a photon (wavy line). (b) shows a simple 2→2 scattering process, also known as a four-point interaction. Panel (c) shows a very a general four-point interaction where the blob in the middle represents an unknown sequence of interactions.*

It turns out that QED has exactly one *fundamental* interaction between two electrons and a photon. This is visualised in panel (a) of the figure where time is taken to run upwards for definiteness and the diagram can be interpreted as the emission of a photon by an electron. As the emitted photon carries both energy and momentum, the trajectory of the electron is modified. Panel (b) illustrates a more complicated process comprising two instances of the fundamental interaction. It describes two electrons coming together, interacting via the electromagnetic field (i.e. a photon) and going apart again. This is the typical scenario of 2→ 2 *scattering process* with two incoming and two outgoing particles. This is not unlike what is actually performed in particle experiments such as the colliders at CERN in Geneva, where two particles are brought together in a collision and then the particles that come out after collision are observed.[1]

The third panel (c) of Figure 1 illustrates the case that is even closer to experiment, namely where one does not know what happening, but only has information about the incoming and about the outgoing particles: The particles are prepared with specific properties (energy, momentum (roughly direction and speed), possibly intrinsic properties known as polarisation) and similarly data is measured for the outgoing particles.

Since particle physics is a *quantum theory*, there is uncertainty about the exact happenings in the blob of panel (c). In principle anything that can happen, does happen, albeit with different probabilities. The principal object of interest is the *scattering amplitude*

$$\mathcal{A}(p_1, p_2, p_3, p_4) \in \mathbb{C}\,, \tag{1}$$

where p_i for $i = 1, \ldots, 4$ is meant to indicate the specifics of the four particles involved in the scattering process. This is often also referred to as *kinematic data*. The scattering amplitude encodes the *probability* of a scattering process with the given choice of kinematic data taking place. It is only a probability since in quantum theory, outcomes of experiments typically only follow a probability distribution if the experiment is repeated often enough.

The amplitude (1) is a complicated function of the kinematic data and in no realistic physical theory its functional form is known exactly. However, it is possible to obtain approximate answer for a given theory that can then be compared to experiment and help to identify promising physical models.

One important approximation uses building up the amplitude from elementary interactions such as panel (a) of Figure 1. In Feynman diagram terms the quantum principle implies that the blob has to be replaced by the sum of all possible diagrams that can be drawn using the elementary interaction of panel (a), without any restriction on topology. However, there is an ordering principle related to the fact that there is an elementary strength associated to the fundamental interaction in panel (a). One assigns a *coupling constant* g to the elementary three-valent vertex and more complicated diagrams tend to have many occurrences of this coupling constant. If the coupling constant is small, these more complicated diagrams are expected to the less important for the final process. This is the basic idea of *perturbation theory*. Moreover, there is also the notion of relating the number of loops of a diagram to Planck's constant, which is also small, and therefore

[1] The current most powerful particle collider at CERN is called the Large Hadron Collider and brings together not electrons but hadrons that are more complicated composite particles.

higher-loop diagrams are also expected to be small corrections to those without loops, known as tree diagrams.

The perturbative approach posits that the amplitude (1) has an expansion of the form

$$\mathcal{A}(p_1, p_2, p_3, p_4) = \sum_{n\geq 0} g^n \mathcal{A}_n(p_1, p_2, p_3, p_4) + \mathcal{A}^{\text{non-pert.}}(p_1, p_2, p_3.p_4)\,, \qquad (2)$$

where the first, power series term is called the perturbative part of the amplitude and is analytic in the coupling constant g around $g = 0$, while the second, non-analytic term is called the *non-perturbative* term and for example can contain terms of the form $e^{-1/g}$.[2]

1.2 String theory

QFT approaches to interactions have been enormously successful. However, they leave certain aspects related to the very high-energy behaviour open as well as the consistent coupling to gravity at the quantum level. Both issues are potentially resolved in string theory where zero-dimensional point particles are replaced by extended one-dimensional strings. In perturbation theory, instead of drawing one-dimensional lines as in Figure 1 for trajectories, one now has to draw two-dimensional surfaces, more precisely Riemann surfaces with boundaries. We will restrict to closed strings here and some examples of so-called *string world-sheets* (generalising the particle trajectories) are shown in Figure 2. Note that string theory only has one kind of string and the different particle types (electron, photon, ...) are thought to be associated with different vibrational patterns of the string that can oscillate like a string instrument.

An important point is that for mathematical consistency of the resulting equations one normally considers *superstrings* that have an additional property called supersymmetry. Moreover, superstrings (strings henceforth) are required to move in a ten-dimensional space-time, such as $\mathbb{R}^{1,9}$, where the superscript indicates that this is a space with one time and nine space directions, therefore generalising the usual $(1+3)$-dimensional Minkowski space $\mathbb{R}^{1,3}$ of special relativity to higher dimensions. Other ten-dimensional space-times where not all ten directions are infinitely extend, but some can be compact, are possible as well and important for trying to connect string theory to the world of our daily experience. Finding the right type of *string compactification* is, however, a long-standing and open problem. In this contribution we will later encounter the case when the ten-dimensional space-time is of the form $\mathbb{R}^{1,D-1} \times T^{10-D}$ for $3 \leq D \leq 10$ and where T^{10-D} is compact torus of dimension $d = 10 - D$.

Figure 2 shows scattering processes that involve four external string states represented by the four circular boundaries of the surface. As in QFT, there is information about the

[2] The power series has typically zero radius of convergence and is only asymptotic and then intimately related to the non-perturbative terms through the theory of resurgence, see for instance [8].

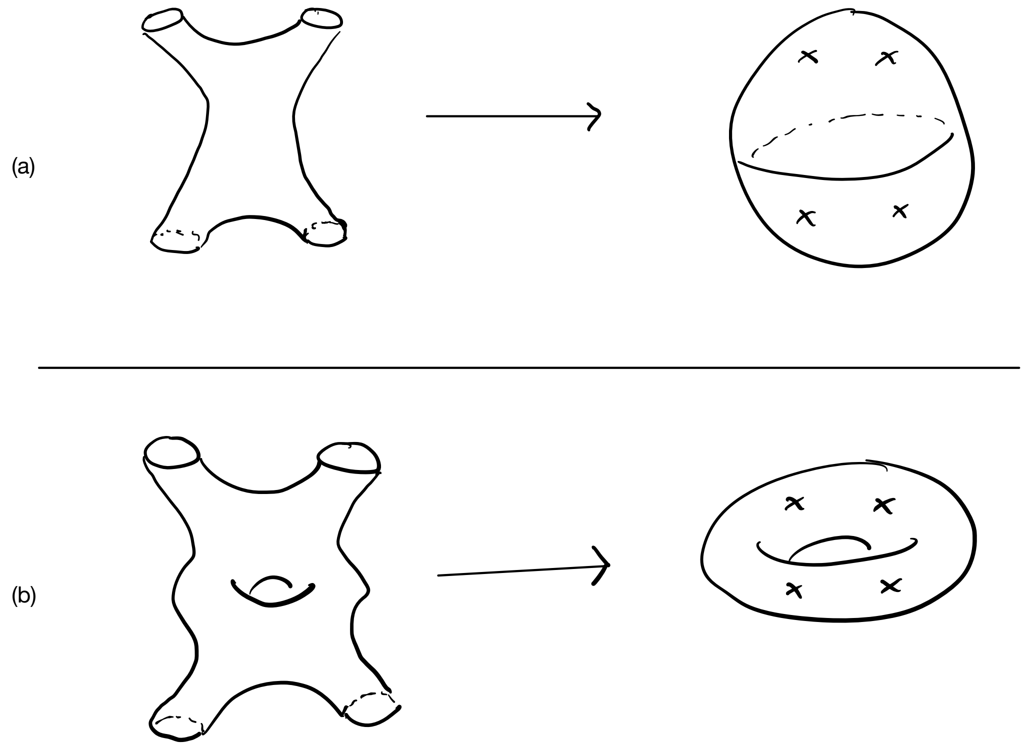

Figure 2: *Two string theory theory world-sheets at genus zero (a) and genus one (b) with four external string states. Going to the right is the result of applying a conformal transformation such that the boundaries are mapped to the marked points (crosses).*

precise kinematic data associated with the boundaries, such as the energy and momentum. String theory is invariant under conformal transformations that can be used to map the string world-sheets to surfaces with marked points as shown when going from the left to the right of the figure. The kinematic data is now at the marked points and in practice implemented via so-called vertex operators.[3]

String theory now associates a string scattering amplitude with a given choice of kinematic data, i.e. string states being scattered. Similar to the perturbative expansion in the coupling constant in (2), there is a perturbative expansion in string theory that involves an expansion in the string coupling constant g_{s}. The power of g_{s} in the expansion is related to the genus of the Riemann surface. Panel (a) of Figure 2 depicts a Riemann surface of genus zero while panel (b) has genus one.

[3]This perturbative picture of string theory is in the language of *conformal field theory*. For textbooks on string theory see [9–11].

The way that the contribution at a given genus is calculated involves an integral over the moduli space $\mathcal{M}_{h,n}$ of Riemann surfaces of genus h with n marked points which is typically hard to perform in practice. One celebrated example where this is possible is for the genus-zero world-sheet with four marked points, where all the marked points are associated with so-called *gravitons*. These are vibrational patterns similar to the basic excitations of the gravitational field and gravitons, just like gravitational waves, have a momentum $k \in \mathbb{R}^{1,9}$ and a polarisation $\epsilon \in S^2(\mathbb{R}^{1,9})$, where S^2 means the symmetric square, so that while k is a vector, ϵ is a symmetric two-tensor.

For the spherical world-sheet of genus zero, string theory gives the contribution [9]

$$\mathcal{A}_{h=0}(k_1,k_2,k_3,k_4;\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_4) = g_{\mathrm{s}}^{-2}\frac{1}{stu}\frac{\Gamma(1-\alpha's)\Gamma(1-\alpha's)\Gamma(1-\alpha's)}{\Gamma(1+\alpha's)\Gamma(1+\alpha't)\Gamma(1+\alpha'u)}\mathcal{R}^4\big(\{k_i\};\{\epsilon_i\}\big) \tag{3}$$

to the four-graviton scattering amplitude. This formula contains the (rescaled) *Mandelstam variables*

$$s = -\frac{1}{4}|k_1+k_2|^2\,, \quad t = -\frac{1}{4}|k_1+k_3|^2\,, \quad u = -\frac{1}{4}|k_1+k_4|^2 \tag{4}$$

that are computed using the Minkowski signature bilinear form $\mathrm{diag}(-,+,+,\ldots,+)$ on $\mathbb{R}^{1,9}$ from pairs of momentum vectors k_i ($i=1,2,3,4$). They are therefore invariant under the Lorentz group $O(1,9)$ that leaves the bilinear form invariant and Lorentz invariance of the amplitude constrains its dependence to the Mandelstam invariants. They moreover satisfy $s+t+u=0$ by conservation of momentum in the scattering process. The quantity $\mathcal{R}^4\big(\{k_i\};\{\epsilon_i\}\big)$ that depends on the momenta k_i and polarisations ϵ_i represents a specific and fixed Lorentz-invariant combination of these objects. Its name arises since it can be thought of as the contraction of four (linearised) Riemann curvature tensors, where the curvature tensor is associated with the gravitational field. Its precise form will play no role in this contribution.

The parameter α' that appears in (3) together with the Mandelstam variables is related to the characteristic length ℓ_{s} of a string by $\alpha' = \ell_{\mathrm{s}}^2$. With this parameter, the combinations $\alpha's$ etc. are dimensionless, meaning that they do not carry any units of length or mass. The Mandelstam variables measure the total energy entering or being transferred in the scattering process. Large amounts of energy (in string units) correspond to $\alpha's \gg 1$. Phenomenologically (and technically) one is often interested in the *low-energy expansion* of an amplitude, that is the regime

$$\alpha's \ll 1, \quad \alpha't \ll 1\,, \quad \alpha'u \ll 1\,. \tag{5}$$

Suppressing the arguments on the left-hand side, the genus-zero amplitude (3) can be expanded for low energies (i.e., for (5)) as

$$\mathcal{A}_{h=0} = g_{\mathrm{s}}^{-2}\mathcal{R}^4\left(\frac{3}{\sigma_3} + (\alpha')^3 2\zeta(3) + (\alpha')^5\zeta(5)\sigma_2 + \ldots\right)\,, \tag{6}$$

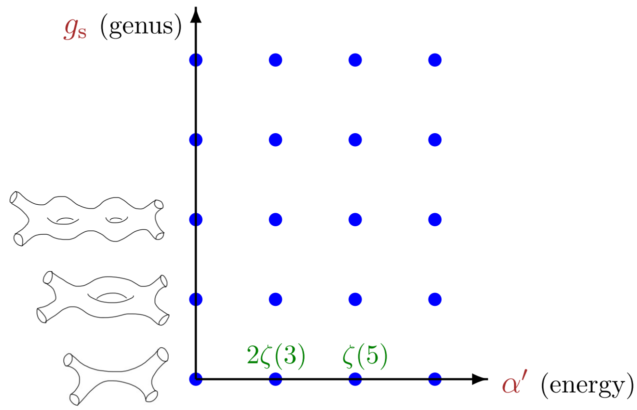

Figure 3: *Schematic depiction of the double expansion of a (perturbative) string scattering amplitude.*

where we have converted to the elementary symmetric polynomials

$$\sigma_2 = s^2 + t^2 + u^2 \,, \quad \sigma_3 = s^3 + t^3 + u^3 \,, \tag{7}$$

which, thanks to $s + t + u = 0$, are the only independent ones and $\zeta(s)$ is the Riemann zeta function.

In summary, we arrive at the picture that a string scattering amplitude has a *double expansion*. The two expansion parameters are the energy scale (often called the α'-expansion) and the genus of the Riemann surface (often called the g_s-expansion). The situation is shown in Figure 3 where two points have been marked with their numerical values as can be deduced from the genus-zero formula (6). Obtaining more numerical values in the figure very rapidly becomes hard at higher genus and we refer the reader to [12] for a recent summary. We stress that both expansions are asymptotic and there are in general non-perturbative terms needed.

The connection to automorphic forms arises at a fixed order in α' but covers all orders in g_s and so corresponds to a vertical line in Figure 3. Remarkably, we will see that the correct interpretation in terms of automorphic forms gives easy access to some higher-genus data and moreover produces the non-perturbative terms that are normally very hard to access.

1.3 Duality symmetries in string theory

In order to understand the occurrence of automorphic forms in string scattering amplitudes we first have to explain what the discrete symmetry giving rise to automorphy is as well as the space that it acts on. The situation will be explained for the simplest case when the string moves in a space-time $\mathbb{R}^{1,D-1} \times T^{10-D}$ known as toroidal compactification of string theory. In this case the discrete symmetry is given by so-called *U-duality* in string theory [13] and the space it acts on is the *moduli space* of such toroidal compactifications which turns out to be a symmetric space for a non-compact Lie group. We will describe these structures in two steps.

String theory on a torus T^d with $d = 10 - D$ depends firstly not only on the geometry of the torus, but also on additional structure that can be put on the torus, namely a two-form (known as B-field in string theory) and a function (known as the dilaton ϕ). The B-field appears since the extended string unlike point particles can also see geometry beyond Riemannian geometry. The geometry and B-field combine such that even and self-dual lattices $II_{d,d}$ of split signature play a role, the definite part of which can be thought of as giving rise to the geometric torus T^d. The moduli space of such lattices is

$$\mathcal{M}_T = O(d,d;\mathbb{Z})\backslash O(d,d;\mathbb{R})/O(d;\mathbb{R}) \times O(d;\mathbb{R})\,, \tag{8}$$

where $O(d,d;\mathbb{Z})$ denotes the discrete group leaving a given lattice $II_{d,d}$ invariant and therefore leading to the same toroidal compactification. This discrete symmetry is known as T-duality in string theory and the fact that string theory is invariant under it can be checked in conformal field theory. The moduli space $\mathcal{M}_T$ is a locally symmetric space associated with the split real Lie group $O(d,d;\mathbb{R})$. The dilaton field mentioned above is invariant under these structures and is actually related to the string coupling constant $g_{\rm s}$ by $g_{\rm s} = e^{\phi}$ (when ϕ is constant, otherwise its expectation value).

Secondly, the toroidal compactification of a string is also sensitive to other data associated with other moduli parameters (known as Ramond–Ramond fields) that extend the above T-duality moduli space $\mathcal{M}_T$ by additional directions. Without explaining the details here, the extra data replaces the locally symmetric space (8) based on $O(d,d;\mathbb{R})$ by one based on $E_{d+1(d+1)}(\mathbb{R})$ [13]:

$$\mathcal{M}_U = E_{d+1(d+1)}(\mathbb{Z})\backslash E_{d+1(d+1)}(\mathbb{R})/K_{d+1}(\mathbb{R})\,, \tag{9}$$

where $E_{d+1(d+1)}$ denotes a split real Lie group in the Cremmer–Julia sequence [14,15] that includes in particular the simply-laced exceptionals. The Dynkin diagram is shown in Figure 4. $K_{d+1}(\mathbb{R})$ is the maximal compact subgroup and the discrete group $E_{d+1(d+1)}(\mathbb{Z})$ is called the U-duality group and is a Chevalley group (that can also be defined as the stabiliser of a suitable lattice of 'charges'). U-duality is a symmetry that is strongly believed to act on any given order in the α'-expression without mixing them [16].

From the Dynkin diagram we see that there is a maximal parabolic subgroup with Levi factor

$$GL(1) \times Spin(d,d;\mathbb{R}) \subset E_{d+1(d+1)}(\mathbb{R}) \tag{10}$$

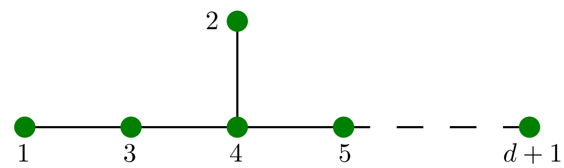

Figure 4: *Dynkin diagram of the Cremmer–Julia sequence of Lie groups of type E_{d+1}. The labelling coming from string theory coincides with that used in Bourbaki.*

where the $Spin(d,d)$ relates back to the lattice $II_{d,d}$ and $GL(1)$ is related to the dilaton ϕ. While the dilaton was invariant under duality transformations from $O(d,d;\mathbb{Z})$, it is no longer invariant under U-duality transformations $E_{d+1(d+1)}(\mathbb{Z})$. Due to the relation $g_s = e^{\phi}$ mentioned above, U-duality therefore mixes different orders in the g_s-expansion and therefore acts along vertical lines in Figure 3. If all the other moduli fields vanish there is in particular a duality transformation that sends

$$\phi \to -\phi \quad \Leftrightarrow \quad g_s \to g_s^{-1} \tag{11}$$

and therefore exchanges small with large values of the string coupling constant g_s.[4] Thus, this transformation is called a strong-weak coupling duality, or S-duality for short. The union of S-duality and T-duality is the full U-duality $E_{d+1(d+1)}(\mathbb{Z})$. Physical quantities in toroidally compactified strings, such as scattering amplitudes, depend on where one is on moduli space $\mathcal{M}_U$.

2 Automorphic forms in scattering amplitudes

As explained in the previous section, a scattering amplitude of a toroidally compactified is a function on $\mathcal{M}_U$, the locally symmetric space introduced in (9). We can reformulate this by introducing functions

$$\begin{aligned} \mathcal{E}_{(p,q)} : E_{d+1(d+1)}(\mathbb{R}) &\to \mathbb{R} \\ \Phi &\mapsto \mathcal{E}_{(p,q)}(\Phi) \end{aligned} \tag{12}$$

for integers $p, q \geq 0$ that are required to satisfy

$$\mathcal{E}_{(p,q)}(\gamma\Phi k) = \mathcal{E}_{(p,q)}(\Phi) \quad \text{for all } k \in K_{d+1} \text{ and } \gamma \in E_{d+1(d+1)}(\mathbb{Z}). \tag{13}$$

[4]This is in fact like the transformation of the imaginary part of a upper half-plane variable under $SL(2,\mathbb{Z})$ which is the particular case of $d = 0$ in the above construction.

These functions appear in the α'-expansion of the (analytic in α'-part of the) string four-graviton scattering amplitude according to

$$\mathcal{A}\big(\{k_i\};\{\epsilon_i\}\big) = \mathcal{R}^4\big(\{k_i\};\{\epsilon_i\}\big)\left(\frac{3}{(\alpha')^3\sigma_3} + \sum_{p,q\geq 0}(\alpha')^{p+q}\mathcal{E}_{(p,q)}(\Phi)\sigma_2^p\sigma_3^q\right). \tag{14}$$

In other words, the functions $\mathcal{E}_{(p,q)}$ summarise the contribution to the amplitude at a fixed order in α', i.e. along a vertical line in Figure 3.

The functions $\mathcal{E}_{(p,q)}$ are the main functions of interest in connection with automorphic forms and automorphic representations. They satisfy

- automorphy and K-invariance, see (13),
- moderate growth in order to be compatible with small g_s expansion for the cusp associated with the parabolic subgroup (10) and similarly at other cusps, and
- differential equations for low values of p and q due to supersymmetry [17–22].

The list of the above properties corresponds to the definition of (spherical) automorphic forms on $E_{d+1(d+1)}$, except for the last condition that does not imply we have a finite-dimensional representation under center $\mathcal{Z}$ of the universal enveloping algebra. Only for $(p,q) = (0,0)$ and $(p,q) = (1,0)$ we will see that the differential equations imply $\mathcal{Z}$-finiteness. We will discuss more general functions further in Section 4.1. In the following we shall write E_{d+1} for brevity.

2.1 Lowest order cases

The first two terms in the α'-expansion correspond to $\mathcal{E}_{(0,0)}$ and $\mathcal{E}_{(1,0)}$ and are called the coefficients of effective interactions known as R^4 and D^4R^4 in physics.

By analysing their differential equations and Fourier expansions they have been identified as (maximal parabolic) Eisenstein series of the type defined by Langlands [23]. Let P_i be the maximal parabolic subgroup of E_{d+1} associated with node i of the Dynkin diagram shown in Figure 4 and

$$\chi_s(p) = \exp\langle 2s\Lambda_i | H(p)\rangle \tag{15}$$

be the character on P_i for $s\in\mathbb{C}$ defined using the fundamental weight Λ_i of the Lie algebra $\mathfrak{e}_{d+1} = \mathrm{Lie}(E_{d+1})$ and H the logarithm map associated with the Levi decomposition of E_{d+1}. More precisely, if $GL(1)$ is the torus part associated with node i and we write an element of E_{d+1} as $\Phi = pk$ with $p\in P_i$ and $k\in K_{d+1}$, then $H(\Phi) = H(p) = \log r$ where $\exp(r)$ is the $GL(1)$ element that is uniquely defined (even though the decomposition of Φ is not).

The character χ_s on P_i can be extended to a map on E_{d+1} since the logarithm map is well-defined on the whole Lie group. Denote by f_{χ_s} the extension of χ_s to E_{d+1}, defined by $f_{\chi_s}(\Phi) = f_{\chi_s}(pk) = \chi_s(p)$. Therefore we can define the *Eisenstein series*

$$E_{i;s}(\Phi) = \sum_{\gamma \in P_i(\mathbb{Z})\backslash E_{d+1}(\mathbb{Z})} f_{\chi_s}(\gamma\Phi)\,. \tag{16}$$

This sum is known to converge for $\mathrm{Re}(s) \gg 1$, more precisely in the Godement range [24] and to satisfy functional relations associated with the Weyl group of E_{d+1} [23]. The functional relations yield analytic continuation to the complex s-plane.

By construction, the Eisenstein series $E_{i;s}$ satisfies $E_{i;s}(\gamma\Phi k) = E_{i;s}(\Phi)$ for $\gamma \in E_{d+1}(\mathbb{Z})$ and $k \in K_{d+1}$. Moreover, the Eisenstein series satisfies differential equations since the character does. Among the simplest one is

$$\left(\Delta - 2s\langle\Lambda_i|s\Lambda_i - \rho\rangle\right) E_{i;s}(\Phi) = 0\,, \tag{17}$$

where Δ is the invariant Laplace operator on the symmetric space E_{d+1}/K_{d+1} corresponding to the quadratic Casimir in the universal enveloping algebra. There are similar differential equations associated with the higher order Casimir operators and $E_{i;s}$ is finite under the center $\mathcal{Z}$.

In terms of the Eisenstein series (16), the two lowest order coefficient functions are [2,3]

$$\begin{aligned} \mathcal{E}_{(0,0)}(\Phi) &= 2\zeta(3)E_{1;3/2}(\Phi)\,, \\ \mathcal{E}_{(1,0)}(\Phi) &= \zeta(5)E_{1;5/2}(\Phi)\,. \end{aligned} \tag{18}$$

The fact that the same numbers already encountered in Figure 3 show up in the normalisations of these functions is not surprising since for $\gamma = 1$ in (16), the values $s = 3/2$ and $s = 5/2$ correspond exactly to the genus-zero contributions to the scattering amplitude. What is surprising and remarkable is that the rest of the scattering amplitude at these orders in α' is simply given by summing over all U-duality images as determined by the Poincaré sum (16).

We note that functional relations allow us to rewrite the functions in (18) in terms of maximal parabolic subgroups different from P_1. These different forms have different physical interpretations, e.g. coming from supergravity or exceptional field theory calculations [25]. The functions $\mathcal{E}_{(p,q)}$ for other values of p and q will be commented on in Section 4.1.

While the specific choices of parameters in (18) have a clear physical explanation, it might seem surprising from a mathematical viewpoint, why some particular choice of parameters should be singled out. It turns out that this can be explained in terms of representation theory which we turn to next.

2.2 Automorphic representations

Given an automorphic form $\eta : E_{d+1} \to \mathbb{C}$, such as the Eisenstein series $E_{i;s}$, one can try to form a representation of the group E_{d+1} by using the right regular action under $g \in E_{d+1}$:

$$\eta \mapsto \pi(g)\eta\,, \quad \text{with} \quad \big(\pi(g)\eta\big)(\Phi) = \eta(\Phi g)\,. \tag{19}$$

As the discrete invariance acts on the left, the new function $\pi(g)\eta$ inherits the same invariance under $E_{d+1}(\mathbb{Z})$. However, it will have different properties with respect to the right-action of K_{d+1}. If the space of functions generated by the right regular action from η

$$\langle \pi(g)\eta | g \in E_{d+1} \rangle \tag{20}$$

satisfies good properties such as finiteness when decomposed with respect to the compact K_{d+1}, the resulting space is called an *automorphic representation*. The precise definition can be found for example in [26, 1] but will not be central here.[5]

Automorphic representations are an important perspective on finding all representations of non-compact Lie groups, such as $E_{d+1}(\mathbb{R})$. The automorphic representations that Eisenstein series belong to are part of the principal series representations of such Lie groups, obtained by induction from characters of parabolic subgroups, very much like the construction in the previous section. We recall that an induced (smooth) representation is defined by

$$\mathrm{Ind}_{P_i}^{E_{d+1}} \chi_s = \{ f \in C^\infty(E_{d+1}) \,|\, f(p\Phi) = \chi_s(p) f(\Phi) \quad \forall p \in P_i \}\ . \tag{21}$$

Whether or not a given representation is unitary depends on the induction parameter(s) s. We note that for the amplitude coefficients $\mathcal{E}_{(p,q)}$ there is no requirement from physics to belong to a unitary principal series representation. The function f_{χ_s} in (16) is a spherical vector in the space $\mathrm{Ind}_{P_i}^{E_{d+1}} \chi_s$, meaning it is right-invariant under K_{d+1}, and it is unique such vector up to scaling.

It is known that principal series representations for higher-rank Lie groups can develop interesting submodules or quotients depending on the value of the induction parameters. Automorphically, this is often referred to through the notion of residues of Eisenstein series. Using this language we can say what is special about the particular functions appearing in (18): They correspond to the induction parameters where the principal series has a non-trivial submodule/subquotient of the smallest (for $\mathcal{E}_{(0,0)}$) or next-to-smallest (for $\mathcal{E}_{(1,0)}$) Gelfand–Kirillov dimension [3, 4]. The first one is known as the *minimal*

[5] Automorphic representations are normally formulated in an adelic setting. The function (16) can be obtained by restricting its corresponding (rational) adelic version to archimedean arguments. Even though this is the proper language for many statements, we will refrain from using it here.

representation of a Lie group and has been studied in the literature, see for example [27] and the next one can be referred to as the *next-to-minimal representation*. We will refer to such representations more generally as *small representations*.

Automorphic forms attached to small representations can be characterized by having very few non-vanishing Fourier coefficients. A simple example that illustrates this is given by theta series. Consider the holomorphic theta series $\theta(\tau) = \sum_{n\in\mathbb{Z}} e^{i\pi\tau n^2}$, where $\tau \in \mathbb{H}$, the complex upper half plane. The square of this theta series famously has a Fourier expansion of the form $\theta(\tau)^2 = \sum_{k=1}^{\infty} R_2(k) e^{2\pi i k\tau}$, where the coefficients $R_2(k)$ count the number of ways in which k can be written as a sum of two squares. Indeed, the only non-vanishing coefficients are those with positive mode number. The fact that so many of the Fourier coefficients vanish is a trademark feature of small representations. In the following we shall discuss Fourier coefficients in the context of more general automorphic forms attached to small representations.

3 Physical information from the Fourier expansion

In the previous section, we have stated in (18) that certain parts of a string theory scattering amplitude can be expressed in terms of special automorphic forms that belong to small representations of the Lie group E_{d+1}. In this section we look at how to extract physical information from this realisation. The key ingredient for this is the Fourier expansion of the automorphic form.

We begin by looking at the simplest case where we can write the full expansion explicitly. This is the case of the rank-one Lie group $E_1 \cong SL(2)$. This is the original case studied in [28] that triggered a large part of the research reported on here.

The symmetric space $SL(2)/SO(2)$ is isomorphic to the Poincaré upper half-plane and we denote the variable on it by $\Omega = \Omega_1 + i\Omega_2$ with $\Omega_2 > 0$. The function $\mathcal{E}_{(0,0)}$ is given by[6]

$$\begin{aligned}\mathcal{E}_{(0,0)}(\Omega) = 2\zeta(3)E_{1;3/2}(\Omega) &= \sum_{\substack{m,n\in\mathbb{Z}\\(m,n)\neq(0,0)}} \frac{\Omega_2^{3/2}}{|m\Omega+n|^3} \\ &= 2\zeta(3)\Omega_2^{3/2} + 4\zeta(2)\Omega_2^{-1/2} + 2\pi\sum_{k\neq 0}\sqrt{|k|}\sigma_{-2}(k)e^{-2\pi|k|\Omega_2+2\pi i k\Omega_1}\left(1+O(\Omega_2)\right),\end{aligned} \tag{22}$$

where we have written out the Fourier expansion associated with the periodicity $\Omega \to \Omega+1$ in the second line. The first two terms correspond to the Fourier zero mode (a.k.a. constant term), while the infinite sum over $k \neq 0$ are the non-zero Fourier modes. Here, $\sigma_s(n) = \sum_{0<d|n} d^s$ is the classical divisor sum.

[6] Since the function is spherical, we work on the quotient $SL(2)/SO(2)$ rather than all of $SL(2)$.

As was mentioned before, there is a relation of the symmetric space variables to physical parameters which here reads

$$\Omega = C_0 + i g_s^{-1} , \tag{23}$$

where C_0 is known as the Ramond–Ramond axion.

Our interest lies first in the dependence on $\Omega_2 = g_s^{-1}$. We see that the Fourier zero mode only contains power-behaved terms of the string coupling. After a suitable conversion[7], one can check that these two terms correspond to the genus-zero and genus-one contributions to the string scattering amplitude at this order in α'. The $2\zeta(3)$ was already indicated in Figure 3 and the $4\zeta(2)$ is associated with the dot just above it since $\mathcal{E}_{(0,0)}$ summarises the amplitude along this vertical axis. The absence of other power-behaved terms means that there are no contributions to the scattering amplitude from higher-genus Riemann surfaces which is a strong prediction from the Eisenstein series and was subsequently understood as a so-called non-renormalisation theorem implied by supersymmetry [29].

The non-zero Fourier modes in (22) all carry the factor

$$e^{-2\pi|k|\Omega_2+2\pi i k\Omega_1} = e^{-2\pi|k|g_s^{-1}+2\pi i k C_0} , \tag{24}$$

which is the characteristic feature of a non-perturbative effect, namely that the Taylor series around weak coupling $g_s \approx 0$ vanishes to all orders. Moreover, the phase coupling to C_0 is also characteristic for what is known as an *instanton effect* and in fact the non-zero modes can be associated with so-called D-instantons in this case [28] that are 'charged' under C_0 and the integer k is known as the instanton charge. The numerical coefficient containing $\sigma_{-2}(k)$ is called the instanton measure and is also of interest in physics and has been investigated using different techniques [30].

The above example shows that the Fourier expansion of $\mathcal{E}_{(p,q)}$ reveals perturbative and non-perturbative information about the scattering amplitude. The $SL(2)$ example is a bit too simple as there is only one cusp ($\Omega \to i\infty$) and one shift symmetry $\Omega \to \Omega + 1$. For higher-rank E_{d+1} there are more choices with a richer structure.

We recall that a Fourier coefficient of an automorphic form η is generally determined by the following data[8]

- a choice of unitary subgroup $U \subset E_{d+1}$ which we take to be the unipotent radical of a parabolic subgroup $Q = LU$; the set of shift symmetries is $U(\mathbb{Z})$, and
- a choice of unitary character $\psi : U \to U(1)$ that is trivial on $U(\mathbb{Z})$; this defines the 'Fourier mode'.

[7] This is related to a so-called change of frame from Einstein to string frame.

[8] More general definitions are possible, but this is sufficient for our purposes here.

The Fourier coefficient is then given by the integral

$$\mathcal{F}_\psi[\eta](\Phi) = \int_{U(\mathbb{Z})\backslash U} \eta(u\Phi)\overline{\psi(u)}du\,. \tag{25}$$

In general, U can be non-abelian and characters ψ are only sensitive to the abelian part $U/[U,U]$. Therefore, the expansion in terms of Fourier coefficients is in general incomplete:

$$\sum_\psi \mathcal{F}_\psi[\eta](\Phi) = \int_{[U,U]} \eta(u\Phi)du\,. \tag{26}$$

Only when U is abelian, the sum of all Fourier coefficients equals the original function η. For this reason, the coefficients (25) are sometimes referred to as *abelian Fourier coefficients*. If one wants to obtain a complete expansion of η one has to supplement them by *non-abelian* Fourier coefficients defined using characters of $[U,U]$ and potentially higher members of the derived series.

In the case of $SL(2)$ with variable Ω the integral in (25) becomes for $\psi(\Omega_1) = e^{2\pi i k\Omega_1}$ the conventional Fourier integral

$$\mathcal{F}_\psi[\eta](\Phi) = \int_0^1 \eta(u\Phi)e^{-2\pi i k\omega}d\omega\,, \tag{27}$$

where

$$\Phi = \begin{pmatrix} \Omega_2^{1/2} & \Omega_2^{-1/2}\Omega_1 \\ 0 & \Omega_2^{-1/2} \end{pmatrix}\,, \quad u = \begin{pmatrix} 1 & \omega \\ 0 & 1 \end{pmatrix} \tag{28}$$

would be standard parametrisations of the $SL(2)$ group elements. Upon evaluating the integral over ω one arrives at the Fourier coefficients for $k \neq 0$ in (22).

3.1 Constant terms

A special and simple case arises for the trivial character $\psi = 1$. This corresponds to the Fourier zero mode and the corresponding Fourier integral (25) in this case is then also known as the constant term (along U). It depends only on the corresponding Levi variables and the most well-studied case is that of the maximal unipotent when the dependence is only on the Cartan torus of the Lie group and in fact for Eisenstein series there are closed formulas due to Langlands that give the result of the constant term integral in terms of intertwining operators [23]. For non-maximal unipotents U similar formulas exist and can be found for example in [31].

These can be applied to the functions given in (18) and the results are given for instance in [2, 1]. Of particular interest from a physical perspective are the following choices of maximal parabolic subgroups $Q = LU$ that can be characterised by which node of the E_{d+1} Dynkin diagram in Figure 4 is removed [32, 2]:

1) Removal of node 1; Levi $GL(1) \times Spin(d,d)$: This connects to string perturbation theory as already anticipated in (10). The constant terms in this case correspond to perturbative string calculations on Riemann surfaces and the non-perturbative terms to D-instantons.

2) Removal of node 2; Levi $GL(d+1)$: This is the M-theory limit and the constant terms correspond to perturbative calculations in toroidally compactified eleven-dimensional supergravity. The non-zero Fourier modes are associated with membrane instantons.

3) Removal of node $d+1$; Levi $GL(1) \times E_d$: This is the decompactification limit where one direction of the torus T^d becomes large and the torus decompactifies to $\mathbb{R} \times T^{d-1}$. The non-zero Fourier modes capture properties of black holes in $D+1$ dimensions whose world-line wraps the special direction.

3.2 Fourier modes and wavefront set

The vanishing properties of the Fourier modes can be neatly summarised by the so-called *wavefront set*, a notion that is familiar from (automorphic) representation theory (see e.g. [33–36]).

Wavefront sets are associated with (automorphic) representations and consist of collections of coadjoint[9] nilpotent orbits in the Lie algebra $\mathfrak{e}_{d+1}$. The connection to Fourier coefficients of automorphic forms is via the unitary character $\psi : U \to U(1)$ entering in (25). Since ψ is defined on a unipotent it can be written on an element $u = \exp(x)$ as

$$\psi(u) = \exp\left(2\pi i\, \omega(x)\right) , \tag{29}$$

where $\omega \in \mathfrak{u}^*$ and, because U is unipotent, the element ω is necessarily nilpotent.

There is an action of the group E_{d+1} on the automorphic form and also on element ω, the vanishing or not of a Fourier coefficient $\mathcal{F}_\psi[\eta]$ depends actually not on ω, but only on the nilpotent orbit $\mathcal{O}_\omega = E_{d+1} \cdot \omega$ it belongs to. We consider the complexification of the orbits and define the wavefront set $WF[\eta]$ as

$$WF[\eta] = \left\{ \text{nilpotent } \mathcal{O}^{\mathbb{C}} \,\middle|\, \exists \omega \in \mathcal{O}^{\mathbb{C}} : \mathcal{F}_\psi[\eta] \neq 0 \right\} . \tag{30}$$

The wavefront set is a property of a given automorphic representation although we have defined it above for one of its members.

Complex nilpotent orbits have a partial order (see e.g. [37]) corresponding to Zariski closure. An interesting question is then what the maximal orbits in the wavefront set of

[9] In all cases considered here, there is a non-degenerate invariant pairing that can be used to identify elements of a Lie algebra with those of its dual.

a given representation are. For irreducible representations it is known that the closely related notion of associated variety is the closure of a *single* nilpotent orbit [38–40]. In terms of the Fourier coefficients this means that there is a single most complicated type of Fourier coefficient for a given automorphic representation.

The constant function has wavefront set given by the trivial orbit. The next possible orbit is the minimal orbit which is unique. An associated representation is called a minimal representation and it is precisely the one associated with the first function $\mathcal{E}_{(0,0)}$ appearing in the string four-graviton amplitude (14). The next bigger orbit is unique for most groups and we call it the next-to-minimal orbit. An associated representation is the one for the function $\mathcal{E}_{(1,0)}$.

The different orbits also have an interpretation in terms of (space-time) supersymmetry in string theory, more precisely, the amount of supersymmetry protection of a given coupling in the effective action. While the supergravity action is fully supersymmetric, the coupling R^4 for the $\mathcal{E}_{(0,0)}$ function is known as half-supersymmetric or $\frac{1}{2}$-BPS. The next one $\mathcal{E}_{(1,0)}$ for the D^4R^4 coupling is $\frac{1}{4}$-BPS. As the number of supersymmetries is finite (=32) one cannot reduce the fraction indefinitely and we shall come back to this point in Section 4.1.

3.3 Reduction algorithm

While the wavefront set records which Fourier coefficients vanish and which ones do not, one might also be interested in the value of the non-vanishing coefficients. In physics, these correspond to properties of instantons as mentioned above or to the counting (modulo sign) of certain black hole microstates in the maximal parabolic 3) of the list in Section 3.1.

Actually carrying out the integral (25) becomes very difficult quickly. It is therefore desirable to reduce it where possible to simpler known integrals. This was addressed in [6,7] where the integral over U was rephrased in terms of sums or integrals of coefficients associated with the maximal unipotent N using a reduction method. In this method the integration domain is extended from U to N in a controlled way such that a finite number of simple operations have to be performed in the process, leading to explicit final formulas.

The interest of the method is that the final answer is expressed in terms of coefficients for the maximal unipotent N which we call *Whittaker coefficients*:

$$\mathcal{W}_\psi[\eta](\Phi) = \int_{N(\mathbb{Z})\backslash N} \eta(ng)\overline{\psi(n)}dn\,. \tag{31}$$

The unitary character ψ on N is not necessarily the same as the original one on U.

Whittaker coefficients (31) can be computed for Eisenstein series η using a reduction formula for degenerate Whittaker vectors [41, 42] that also uses intertwining operators.

For small representations there are again additional simplifications since many terms vanish. Using the reduction algorithm of [6,7] one can show for example that for a minimal automorphic form $\eta_{\rm min}$ and a minimal unipotent character ψ that is only non-zero on a single simple root space one has

$$\mathcal{F}_\psi[\eta_{\rm min}] = \mathcal{W}_\psi[\eta_{\rm min}]\,, \tag{32}$$

i.e., the integration domain and unitary character can be extended with impunity. For all the groups E_{d+1} the above coefficients have the same simple structure due to minimality. Indeed, the Whittaker coefficient $\mathcal{W}_\psi[\eta_{\rm min}]$ takes the form of a single divisor sum $\sigma_s(k)$ multiplied by a K-Bessel function. This mimics the structure of the Fourier coefficients of the $SL(2)$-Eisenstein series (22).

Similarly, for a next-to-minimal automorphic form $\eta_{\rm ntm}$ and a unitary character ψ on U of rank two (Bala–Carter type $2A_1$) one can find an element $\gamma \in E_{d+1}$ such $\mathrm{Ad}^*_\gamma\psi = \psi_{2A_1}$ with ψ_{2A_1} being supported on two orthogonal simple root spaces. Then

$$\mathcal{F}_\psi[\eta_{\rm ntm}](\Phi) = \int_{V_\gamma} \mathcal{W}_{\psi_{2A_1}}[\eta_{\rm ntm}](v\gamma\Phi)dv\,, \tag{33}$$

where V_γ is an adelic integration that is determined by U and γ. Again, for all the groups E_{d+1} these coefficients have a simple structure, that generalizes the minimal case. The coefficients are all of the schematic form $\sigma_s(k)\sigma_{s'}(k')\int KK$, i.e. they involve a product of divisor functions along with an integral over two K-Bessel functions [42, 7].

In general, the formula is expressed as an adelic integral, and as such it maps Eulerian coefficients to Eulerian coefficients [43].

4 Outlook

In this section, we discuss some of the many fascinating open problems. The reader is again invited to consult [1–7] as well as the references given below for more details.

4.1 Beyond $\mathcal{Z}$-finiteness

The first obvious point that we would like to address is what happens when going further in the low-energy expansion, i.e., what happens for the next vertical lines in Figure 3, or, equivalently, for the functions $\mathcal{E}_{(p,q)}$ with values of (p, q) different from those in (18).

From physical supersymmetry considerations one expects these functions to obey differential equations that are qualitatively different from the $\mathcal{Z}$-finiteness conditions like (17). For example, it has been known for some time that in the case of $E_1 \cong SL(2)$ the function $\mathcal{E}_{(0,1)}$ satisfies [19]

$$(\Delta - 12)\mathcal{E}_{(0,1)} = -\mathcal{E}^2_{(0,0)}\,, \tag{34}$$

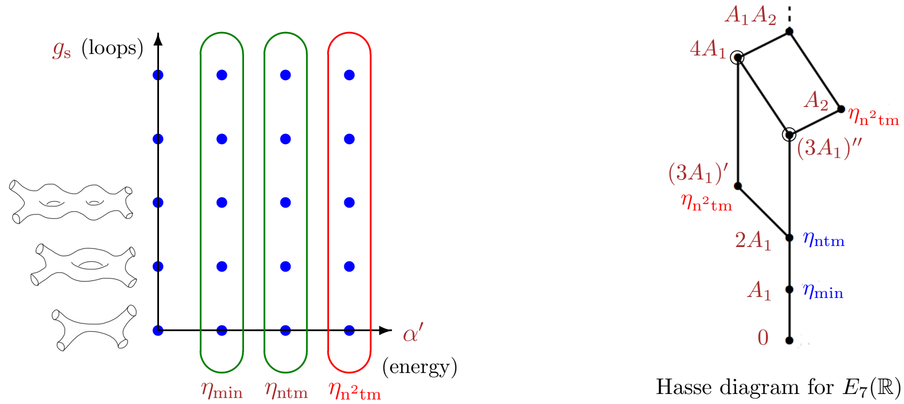

Figure 5: *Some nilpotent orbits in relation to terms in the string scattering amplitude. The circled orbits in the Hasse diagram are not special.*

where importantly the *square* of the (minimal) function $\mathcal{E}_{(0,0)}$ appears on the right-hand side of the equation. Similar equations for higher rank have been given in [22].

Acting multiple times with the Laplacian on $\mathcal{E}_{(0,1)}$ generates a non-finite space, since the square $\mathcal{E}^2_{(0,0)}$ is not in a finite-dimensional representation of $\mathcal{Z}$. Therefore, the corresponding automorphic form cannot belong to a standard automorphic representation since one of the standard defining properties ($\mathcal{Z}$-finiteness) is violated.

Nevertheless, it has been possible to solve (34) for $SL(2)$ [5] and also for the general E_{d+1} case [44, 25, 45] and generalisations of such inhomogeneous Laplace equations have attracted quite some interest and approaches with different methods [46–49].

Instead of displaying the solutions here which would require introducing quite some notation, we comment on interesting qualitative features. Since the function is not $\mathcal{Z}$-finite, it must belong to a novel type of automorphic representation. How to define this in a useful generality is currently an open problem, but examples similar to the one above from string theory can be important guiding beacons.

One aspect where they clearly differ from the standard representation is in the wavefront set. This is displayed in Figure 5 where in the left panel the function $\mathcal{E}_{(0,1)}$ is indicated by a red oval labeled $\eta_{\rm n^2tm}$ for the next function after the next-to-minimal one. In the right panel is the Hasse diagram of nilpotent orbits for the case of E_7 that is useful for indicating the wavefront set and the following statements are restricted to this case. For the minimal and next-to-minimal case there is a single maximal orbit in the wavefront set, corresponding to the orbits of Bala–Carter types A_1 and $2A_1$, respectively. By contrast, there are two maximal orbits for $\eta_{\rm n^2tm}$, namely $(3A_1)'$ and A_2 that appear because of the physical boundary conditions. This type of behaviour is not possible for standard automorphic representations, but consistent with physical expectations where the two orbits correspond to two distinct types of $\frac{1}{8}$-BPS states.

A further intriguing feature of the solution is that the numerical value of some of the explicit Fourier coefficients grows exponentially with the instanton charge, i.e., the norm of the nilpotent element of the U-duality orbit. This is also in contrast with standard behaviour of Fourier coefficients. Again, the exponential growth is actually expected from physics where it relates to the counting of black hole microstates and the exponential behaviour is required for obtaining a non-zero entropy.

We close these observations by pointing out that the reduction algorithm of [6] is valid for functions irrespective of whether or not they are $\mathcal{Z}$-finite. As the final result is in terms of (degenerate) Whittaker coefficients that are only easy to determine for Eisenstein series, the method is however not directly applicable to the end.

4.2 Further directions

We loosely list some additional interesting topics.

- The construction above relied heavily on toroidal compactifications of string theory. While these have the most rigid mathematical structure, they are the least realistic ones. A compactification scenario from ten to four space-time dimensions that has been studied extensively is that where the torus T^6 is replaced by a complex Calabi–Yau threefold. How much of the nice story of automorphic representations and couplings survives in this case is not clear, although likely very little. It has been proposed that at least in the setting of rigid Calabi–Yau threefolds, the relevant automorphic forms would be attached to the so called quaternionic discrete series [50–52, 1]. In recent years, the mathematical understanding of automorphic forms attached to the quaternionic discrete series has been greatly developed due to the works of Pollack and collaborators [53–56], following the original paper of Gan, Gross and Savin [57].

- There are typically L-functions associated with automorphic forms and these enter crucially in the Langlands conjectures. What is the physical interpretation of these L-functions and do they exist also for the generalised automorphic forms starting with $\mathcal{E}_{(0,1)}$? A possible physical interpretation of L-functions in the context of mirror symmetry has been proposed in [58–60]. It would be very interesting to understand whether there is any relation to the automorphic structures discussed in this survey.

- Is there an extension of the structures to the Kac–Moody case? Are there similar wavefront sets and small representations? First investigations seem to point in this direction [61, 42]. It is well-known that due to the absence of a longest Weyl word for Kac–Moody groups, the generic Whittaker coefficients identically vanish for Kac–Moody groups [62]. The interesting Fourier coefficients are therefore degenerate Whittaker coefficients, i.e. those for which the unitary character ψ_N is only supported on a subset of the simple roots of E_{d+1}. It was shown in [42] that for special values of s the only non-vanishing Whittaker coefficients of Eisenstein series on the Kac–Moody groups E_9, E_{10}, E_{11} are those for which ψ_N is only non-trivial on a single simple roots. This indicates that all Fourier coefficients are completely determined by maximally degenerate Whittaker coefficients. This is a telltale sign of a minimal representation. It would be very interesting to investigate this further as a window into a possible theory of small representations of Kac–Moody groups.

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